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Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

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PARAMETRIC SURFACES (IV)
THE INTEGRAL FORMULA FOR THE AREA

By A. S. BESICOVITCH (*Cambridge*)

[Received 2 March 1948]

INCONSISTENCY of the Lebesgue–Frechet definition of the area of a surface with fundamental ideas concerning the area† has made it necessary to adopt a new definition. In the light of the modern theory of functions of a real variable the area should be considered as a measure and I define it as a Carathéodory–Hausdorff two-dimensional measure.

This raises anew the question of expressing the area in the form of an integral. In the first of the papers quoted in the footnote the problem was solved for the Tonelli case of surfaces defined by an equation of the form $z = f(x,y)$. In the present paper a solution is given for the general case of surfaces defined parametrically.

Two-termed and three-termed column matrices

$$s = \begin{vmatrix} u \\ v \end{vmatrix}, \qquad x = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

will denote points in a plane and in the space. When the coordinates of the points are inessential, I shall denote them also by single letters M, P.

Let $x = \Phi(s)$, where s varies on a disc H of radius unity, be a continuous function. Any saturated continuum Q in H on which x is constant is called a Φ-*element*. Thus $\Phi(s)$ establishes a representation of H as a sum of different Φ-elements. A Φ-element may be, of course, a single point. We form the set of pairs

$$\Pi = \sum (P, Q),$$

where Q runs through all Φ-elements of H and, for any Q, P is the value of $\Phi(s)$ on Q. The set Π is called a *parametric surface*, and pairs (P, Q) are *points of the parametric surface*. Two points (P, Q) and (P', Q') are *identical* only if $Q = Q'$ (and then, of course, $P = P'$). If there are exactly k different points of Π with the same P, then

† A. S. Besicovitch, 'On the definition and value of the area of a surface', *Quart. J. of Math.* (Oxford) 16 (1945), 86–102. A. S. Besicovitch, 'Parametric Surfaces (II). Lower semi-continuity of the area', *Proc. Cambridge Phil. Soc.* (in press).

we say that P *is of multiplicity* k on Π. By $\Phi(s)$ we mean two different things: either it is the point $x = \Phi(s)$ or it is the point (P, Q) of the parametric surface, where Q is the Φ-element containing s. As it will always be clear from the text which meaning is attributed, we do not use different notations. Let now G be a sub-set of H; then by $\Phi(G)$ we mean the set of *points* $\Phi(s)$ *of the parametric surface* for all $s \in G$. Thus $\Phi(G)$ is the set of those pairs (P, Q) for which Q either belongs to G or has, at least, one point in common with G. If there are exactly k' points of $\Phi(G)$ with the same P, we say that P is of *multiplicity* k' on $\Phi(G)$. By $|\Phi(G)|$ we mean the set of different P belonging to points of $\Phi(G)$. Thus $\Phi(G)$ is a set of points of the parametric surface, while $|\Phi(G)|$ is a set of points in the three-dimensional space. Write

$$|\Phi(G)| = \sum E_k,$$

where the summation is extended over all positive integral values of k and the value $k = \infty$, and E_k is the set of those points of $|\Phi(G)|$ that have multiplicity k on $\Phi(G)$.

The Carathéodory Λ^2-measure of $\Phi(G)$ is defined by the equation

$$\Lambda^2\Phi(G) = \sum k\Lambda^2 E_k,$$

where the term corresponding to $k = \infty$ is equal to zero, if $\Lambda^2 E_\infty = 0$, and to ∞ otherwise. $\Lambda^2\Phi(H) = \Lambda^2\Pi$ is called the *area of the parametric surface.*

If $\Lambda^2\Phi(G) = 0$ whenever $\Lambda^2 G = 0$, we say that the function $\Phi(s)$ is *absolutely continuous.*

I shall denote the approximate or the exact partial derivatives of $x = \Phi(s)$ with respect to u and to v by the symbols

$$a(s) = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix}, \qquad b(s) = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

and I shall write

$$D(s) = +\sqrt{\left\{\begin{vmatrix} a_2, & b_2 \\ a_3, & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3, & b_3 \\ a_1, & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2\right\}}.$$

The purpose of this paper is to prove the

THEOREM. *If* $\Phi(s)$ *is absolutely continuous and approximately differentiable at almost all points* (*in the sense of* Λ^2*-measure*) *of* H, *then*

$$\Lambda^2\Pi = \iint_H D(s)\, dudv,$$

whether the integral is finite or infinite.

By the Lusin theorem on continuous functions and by the Stepanoff† theorem on differentiability of functions of two variables, given $\alpha > 0$ we can represent H in the form

$$H = H_1 + H_2 + H_3,$$

where

(i) $\Lambda^2 H_3 < \alpha$;

(ii) $\Phi(s)$ is uniformly totally differentiable on the whole of $H_1 + H_2$, and the partial derivatives are continuous and bounded;

(iii) $D(s)$ is zero or positive according as $s \in H_1$ or $s \in H_2$.

LEMMA 1. $$\Lambda^2 \Phi(H_1) = 0.$$

Take a small $\rho_0 > 0$ and define a set $\Gamma(\rho_0)$ of disjoint circles of radii not exceeding ρ_0 in H, satisfying the conditions

(i) the centre of each circle is in H_1;

(ii) $\Lambda^2\{H_1 - \Gamma(\rho_0)\} = 0$.

Let $$c(s_i, \rho_i), \qquad s_i = \begin{vmatrix} u_i \\ v_i \end{vmatrix}, \qquad \rho_i < \rho_0 \qquad (i = 1, 2, \ldots)$$

be the circles of $\Gamma(\rho_0)$; we have $\sum \rho_i^2 \leqslant 1$. For any $s \in H_1 c(s_i, \rho_i)$ we have

$$x(s) - x(s_i) = a(s_i)(u - u_i) + b(s_i)(v - v_i) + o(\rho_i).$$

Since $D(s_i) = 0$, one of the columns $a(s_i)$, $b(s_i)$ is a scalar multiple of the other, say

$$b(s_i) = ka(s_i) \quad (-1 \leqslant k \leqslant +1).$$

We have

$$x(s) - x(s_i) = a(s_i)\{(u - u_i) + k(v - v_i)\} + o(\rho_i),$$
$$-2\rho_i \leqslant (u - u_i) + k(v - v_i) \leqslant 2\rho_i.$$

Thus all points of $|\Phi\{H_1 c(s_i, \rho_i)\}|$ are within $o(\rho_i)$ of some vector $\lambda a(s_i)$, where λ is a scalar such that $-2\rho_i \leqslant \lambda \leqslant 2\rho_i$. Since $a(s)$ is bounded on $H_1 + H_2$, all the points of $|\Phi\{H_1 c(s_i, \rho_i)\}|$ can be included in a set U_i of spheres such that

$$\sum_{U_i} r^2 = o(\rho_i^2).$$

Writing $U = \sum U_i$ we have

$$|\Phi\{H_1 \Gamma(\rho_0)\}| \subset U,$$

and $$\sum_{U} r^2 = o\left(\sum \rho_i^2\right) = o(1).$$

† W. Stepanoff, 'Sur les conditions de l'existence de la différentielle totale', *Rec. Math. Soc. Math. Moscou*, 32 (1925), 511–26.

Hence, by (ii) and by the absolute continuity of $\Phi(s)$, we have

$$\Lambda^2|\Phi(H_1)| = 0,$$

and consequently $$\Lambda^2\Phi(H_1) = 0.$$

This completes the proof.

Lemma 2. *Given a small $\epsilon > 0$ and two sets A, A' of points in the three-dimensional space, and a one-to-one correspondence $P \sim P'$ between the points of the sets such that*

$$P_1 \sim P_1', P_2 \sim P_2' \quad \textit{implies} \quad \overrightarrow{P_1'P_2'} = \overrightarrow{P_1P_2}+\epsilon|\overrightarrow{P_1P_2}|\theta$$

where θ is a vector of length $\leqslant 1$, and $|\overrightarrow{P_1P_2}|$ is the length of $\overrightarrow{P_1P_2}$, then

$$\Lambda^2A' = (1+5\phi\epsilon)\Lambda^2A \quad (-1 < \phi < +1).$$

The lemma follows at once from the fact that, if some part of A is included in a sphere of diameter d, then the corresponding part of A' can be included in a sphere of diameter $(1+2\epsilon)d$ and vice versa.

Lemma 3. *If at the point $s_0 = \begin{vmatrix} u_0 \\ v_0 \end{vmatrix}$ of H_2*

$$\frac{\partial x}{\partial u} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \qquad \frac{\partial x}{\partial v} = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix},$$

then $$\Lambda^2\Phi\{H_2\,c(s_0,r)\} = \{1+o\,(1)\}\Lambda^2\{H_2\,c(s_0,r)\} \tag{1}$$

for small r.

For let $s' = \begin{vmatrix} u' \\ v' \end{vmatrix}$, $s'' = \begin{vmatrix} u'' \\ v'' \end{vmatrix}$ be any pair of points of $H_2\,c(s_0,r)$. By the continuity of partial derivatives on H_2 and by the uniform differentiability we have

$$x_1(u'',v'')-x_1(u',v') = u''-u'+o\,(|u''-u'|+|v''-v'|),$$
$$x_2(u'',v'')-x_2(u',v') = v''-v'+o\,(|u''-u'|+|v''-v'|),$$
$$x_3(u'',v'')-x_3(u',v') = o\,(|u''-u'|+|v''-v'|).$$

Hence, if $$x' = \Phi(s'), \qquad x'' = \Phi(s''),$$

$$\overrightarrow{x'x''} = \overrightarrow{s's''}+o\,(|\overrightarrow{s's''}|)\theta,$$

where θ is a vector of length 1. Thus $\Phi\{H_2\,c(s_0,r)\}$ has no multiple points, so that

$$\Lambda^2\Phi\{H_2\,c(s_0,r)\} = \Lambda^2|\Phi\{H_2\,c(s_0,r)\}|.$$

The result follows from Lemma 2.

Remark. It is obvious that the formula (1) remains true if we replace $H_2 c(s_0, r)$ by any neighbourhood H' of s_0 in H_2, of small diameter.

LEMMA 4. *If at a point* $s_0 = \begin{vmatrix} u_0 \\ v_0 \end{vmatrix}$ *of* H_2

$$\frac{\partial x}{\partial u} = a = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix}, \qquad \frac{\partial x}{\partial v} = b = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix},$$

then $$\Lambda^2\Phi\{H_2 c(s_0, r)\} = \{1+o(1)\}D(s_0)\Lambda^2\{H_2 c(s_0, r)\}$$

for small r.

For let $x^0 = \Phi(s_0)$. Change coordinates by the formula

$$x' = A(x-x^0),$$

where A is a square matrix such that $|A| = 1$, taking x^0 for the new origin and the tangential plane at x_0 for the plane $x_3' = 0$ and transform $s = \begin{vmatrix} u \\ v \end{vmatrix}$ into $s' = \begin{vmatrix} u' \\ v' \end{vmatrix}$ by the formula $s' = B(s-s_0)$, where B is a non-singular square matrix of two lines, so that, at $s' = 0$,

$$\frac{\partial x'}{\partial u'} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \qquad \frac{\partial x'}{\partial v'} = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}.$$

Then it is easy to see that the Λ^2-measure of the transform of $H_2 c(s_0, r)$ is equal to $D(s_0)\Lambda^2\{H_2 c(s_0, r)\}$ and, by the remark to Lemma 3,

$$\Lambda^2\Phi\{H_2 c(s_0, r)\} = \{1+o(1)\}D(s_0)\Lambda^2\{H_2 c(s_0, r)\}.$$

LEMMA 5. $$\Lambda^2\Phi(H_1+H_2) = \iint\limits_{H_1+H_2} D(s)\, dudv.$$

Proof. Given $\epsilon > 0$ and $\rho_0 > 0$, define a positive function $\rho(s) \leqslant \rho_0$ on H_2 satisfying the conditions

(i) all points of $\Phi[H_2 c\{s, \rho(s)\}]$ are simple;

(ii) for any $s' \in H_2 c\{s, \rho(s)\}$

$$\frac{D(s')}{D(s)} = 1+\tfrac{1}{3}\theta\epsilon \quad (-1 < \theta < +1);$$

(iii) for any $r \leqslant \rho(s)$

$$\Lambda^2\Phi\{H_2 c(s, r)\} = (1+\tfrac{1}{3}\theta\epsilon)D(s)\Lambda^2\{H_2 c(s, r)\},$$

and thus by (ii)

$$\Lambda^2\Phi\{H_2\,c(s,r)\} = (1+\theta\epsilon)\iint\limits_{H_2 c(s,r)} D(s')\,du\,dv.$$

Denote by Γ the set of circles $c(s,r)$ for all $s \in H_2$ and $r \leqslant \rho(s)$. By the Vitali principle there exists a sub-set Γ' of disjoint circles of Γ such that

$$\Lambda^2(H_2-\Gamma') = 0. \tag{2}$$

By (iii) and (2)

$$\sum_{\Gamma'} \Lambda^2\Phi\{H_2\,c(s,r)\} = (1+\theta\epsilon)\iint\limits_{H_2} D(s)\,du\,dv. \tag{3}$$

Denote by B the set of the circumferences of the circles of Γ'. Obviously $\Lambda^2 B = 0$ and $\Lambda^2\Phi(B) = 0$. The same Φ-element, if it differs from a point, may have points in common with more than one circle of Γ', but then it would also have points in common with B. Thus members of the sum

$$\sum_{\Gamma'} \Phi\{H_2\,c(s,r)\}$$

may have points in common, but, as all such points belong to $\Phi(B)$, they form a set of Λ^2-measure zero. Hence

$$\sum_{\Gamma'} \Lambda^2\Phi\{H_2\,c(s,r)\} = \Lambda^2 \sum_{\Gamma'} \Phi\{H_2\,c(s,r)\} = \Lambda^2\Phi(H_2\,\Gamma') = \Lambda^2\Phi(H_2),$$

and by (3)
$$\Lambda^2\Phi(H_2) = (1+\theta\epsilon)\iint\limits_{H_2} D(s)\,du\,dv.$$

This being true for any $\epsilon > 0$ and H_2 being independent of ϵ, we have

$$\Lambda^2\Phi(H_2) = \iint\limits_{H_2} D(s)\,du\,dv,$$

and, by Lemma 1,

$$\Lambda^2\Phi(H_1+H_2) = \iint\limits_{H_1+H_2} D(s)\,du\,dv,$$

since $D(s) = 0$ on H_1.

Proof of the theorem. If $\iint\limits_{H} D(s)\,du\,dv = \infty$, then $\iint\limits_{H_1+H_2} D(s)\,du\,dv$ may be as large as we please, since $\Lambda^2 H_3$ may be as small as we please. Hence $\Lambda^2\Pi = \infty$.

If
$$\iint\limits_{H} D(s)\,du\,dv < \infty,$$

then from Lemma 5 and from the absolute continuity of $\Phi(s)$ it follows in the usual way that

$$\Lambda^2\Pi = \iint_H D(s)\, dudv.$$

This proves the theorem.

The method of the proof of the theorem shows that, if $\Phi(s)$ satisfies the conditions of the theorem and if G is any measurable sub-set of H, then

$$\Lambda^2\Phi(G) = \iint_G D(s)\, dudv.$$

CONVERGENCE OF FAMILIES OF COMPLETELY ADDITIVE SET FUNCTIONS

By J. L. B. COOPER (*London*)

[Received 29 January 1948]

1. Introduction

A NUMBER of theorems exist which give necessary conditions for the convergence of families of linear operators; often the proofs follow very similar patterns, and this suggests the possibility of coordinating the particular theorems in a general statement. Such a general theorem will be given in this note, together with an investigation of how far the hypotheses can be relaxed, and indications of applications which generalize and simplify the proofs of a number of known theorems.

Theorems on families of linear functionals fall into two main types. The first states that under certain conditions families of operators bounded at each point of a space are bounded everywhere [e.g. Hahn (**1**), Theorem I]. Very general theorems of this sort can be deduced from the Baire category theorem [cf. (**2**) 19, Theorem 11]. The second type, with which this paper deals, states circumstances under which a family, which converges everywhere in a space, converges uniformly or perhaps tends to zero uniformly. Such theorems apply only to more special types of spaces: essentially they apply to operators on a completely additive Boolean ring, or to cases easily reducible to this. On the other hand, the spaces involved need not be metricizable. The proof given here is of a more constructive type than are those depending on the category theorem.

2. Terminology and notation

I shall deal with families of completely additive set functions (abbreviated to c.a.s.f.) $L_i(X) \equiv L(i; X)$ defined for all indices i in a set of indices **I** and for all sets X in a family of sets **K** which satisfies the conditions:

K 1. The union and the intersection of any finite or countable set of sets of **K** belongs to **K**.

K 2. The difference of any two sets of **K**, one of which contains the other, lies in **K**.

It is not necessary to assume that the complement of a set of **K** lies in **K**. (A counter-example is the set of countable sub-sets of a

non-countable set.) It is also unnecessary to suppose that the elements of **K** are sub-sets of some set. (As a counter-example, each element of **K** could be the set of all Lebesgue measurable sub-sets of $(0, 1)$ differing by a set of measure zero from a given sub-set; **K** is the ring of Lebesgue measurable sets *modulo* sets of zero measure.) In view of this, and to avoid repetition of the word 'set' I shall refer to **K** as a *complete Boolean ring*, and call its members $X, Y, \ldots$ *elements*.

The assumptions made about **K** may be put, in abstract form, as follows:

K′ 1. **K** is a partially ordered set, ordered by a relation (inclusion), denoted by $\subset$.

K′ 2. Any finite or countably infinite aggregate of elements of **K** has a least upper bound, denoted by $X \cup Y \cup Z \cup \ldots$ or $\bigcup X_\alpha$, and called the *union of the set of elements*, and a greatest lower bound, denoted by $X \cap Y \cap Z \ldots$ or $\bigcap X_\alpha$ and called the *intersection of the set of elements*.

K′ 3. Union and intersection obey the distributive laws:

$$X \cup [\bigcap Y_\alpha] = \bigcap [X \cup Y_\alpha], \qquad X \cap [\bigcup Y_\alpha] = \bigcup [X \cap Y_\alpha].$$

K′ 4. **K** has a least member, denoted by ϕ.

K′ 5. If $X \subset Y$, there is a unique element Z in **K** such that

$$X \cup Z = Y, \qquad X \cap Z = \phi.$$

Z is called the *difference* between Y and X: $Z = Y - X$.

Elements $X, Y \in \mathbf{K}$ such that $X \cap Y = \phi$ are called *disjoint*.

A c.a.s.f. (completely additive set function) on **K** is a function $L(X)$ defined and with a real finite value for each $X \in \mathbf{K}$, and such that, if $\{X_n\}$ is any sequence of mutually disjoint elements of **K** with union X, then

$$L(X) = \sum L(X_n).$$

The total variation of L on X, that is to say, the difference between the upper and lower bounds of $L(Y)$ for all elements $Y \subset X$, is also a c.a.s.f. [cf. (**3**) 10–11] and will be denoted by $|L|(X)$.

For convenience I shall give some definitions connected with the notion of a *filter*, due to H. Cartan [(**4**) and (**5**) 20] which is extremely useful for a general discussion of limiting processes.

A *filter* on a set E is a set $\mathfrak{F}$ of sub-sets of E such that:

F 1. If $F_1 \in \mathfrak{F}$ and $F_2 \supset F_1$, then $F_2 \in \mathfrak{F}$;

F 2. If $F_1, F_2 \in \mathfrak{F}$, then $F_1 \cap F_2 \in \mathfrak{F}$;

F 3. No set in $\mathfrak{F}$ is empty.

A set of sub-sets of E, $\mathfrak{B}$, is said to form a *base* of the filter $\mathfrak{F}$ if, for any $F \subset E$, $F \in \mathfrak{F}$ if and only if there is a set $B \in \mathfrak{B}$ such that $B \subset F$ [see (**5**) 23].

If $f(x)$ is a function of elements x of E, taking its values in some topological space, $f(x)$ is said to *tend* to a point y along the filter $\mathfrak{F}$ if for any neighbourhood V of y there is a set $F \in \mathfrak{F}$ such that $f(x) \in V$ if $x \in F$. We then write $\lim_{\mathfrak{F}} f(x) = y$.

In this article we are concerned with families of c.a.s.f. $\{L_i\}$ defined on a complete Boolean ring $\mathbf{K}$ for all indices i in a set $\mathbf{I}$. On the one hand, we shall consider limits of $L_i(X)$ for varying i, on the other, limits for varying X. In the first case I suppose that there is given some filter $\mathfrak{F}$ on $\mathbf{I}$, and consider limits along this filter. The second case, that of varying X, needs a new concept, which I introduce with the aim of generalizing the notion of a c.a.s.f. absolutely continuous with respect to a measure function. The concept is defined as follows:

A filter $\mathfrak{M}$ on a complete Boolean ring is called a *mesh** if the filter has a base $\mathfrak{B}$ such that, if $X \in B \in \mathfrak{B}$ and $Y \subset X$, then $Y \in \mathfrak{B}$.

The most important special case is that in which a measure μ is defined on a complete Boolean ring $\mathbf{K}$, and the base $\mathfrak{B}$ of the mesh is taken to consist of all the sets of form B_σ, where σ runs through all positive real numbers, and B_σ is the set of all elements of K whose measure is less than σ. In this case we say that the mesh is *metricizable*, or that it is deduced from the measure.

A c.a.s.f. L defined on a complete Boolean ring is said to be *absolutely continuous* with respect to a mesh $\mathfrak{M}$ if for every $\delta > 0$ there is a set $M \in \mathfrak{M}$ such that $L(X) < \delta$ if $X \in M$. It is obviously equivalent to this to require that for each $\delta > 0$ there should be an $M \in \mathfrak{M}$ such that $|L|(X) < \delta$ if $X \in M$.

For a mesh deduced from a measure, the definition of absolute continuity coincides with the classical definition. For such a mesh the intersection of all sets of the mesh consists of all the elements of zero measure, and a c.a.s.f. is absolutely continuous if and only if it is zero for all elements of zero measure. Similarly, if we call an element which lies in the intersection of all the sets of a mesh a *zero element*, then a c.a.s.f. which is absolutely continuous with respect to the mesh is zero for each zero element, but the converse is not true.

* A general theory of meshes will be developed in a later paper.

3. THEOREM 1. *Let $L_i(X) = L(i; X)$ be a family of c.a.s.f. each defined for all elements X of a complete Boolean ring* $\mathbf{K}$, *and absolutely continuous with respect to a mesh* $\mathfrak{M}$ *on* $\mathbf{K}$; *for each X in* $\mathbf{K}$ *let $L_i(X)$ tend to a limit according to a filter* $\mathfrak{F}$ *on the set* $\mathbf{I}$ *of indices* $\{i\}$, *where the filter* $\mathfrak{F}$ *has a countable base. Then* (A) *for all $\delta > 0$ there exists a set* $\mathbf{M}(\delta)$ *in* $\mathfrak{M}$ *and* $\mathbf{F}(\delta)$ *in* $\mathfrak{F}$ *such that*

$$|L_i|(X) < \delta \quad \textit{if} \quad X \in \mathbf{M} \quad \textit{and} \quad i \in \mathbf{F}.$$

It is obvious that the condition (A) is equivalent to the condition (A′) *for all $\delta > 0$ there exists* $\mathbf{M}'(\delta) \in \mathfrak{M}$ *and* $\mathbf{F}'(\delta)$ *in* $\mathfrak{F}$ *with*

$$|L(X)| < \delta \quad \textit{if} \quad X \in \mathbf{M}' \quad \textit{and} \quad i \in \mathbf{F}'.$$

It is clear that, if (A) is satisfied, (A′) is satisfied with $\mathbf{M}'(\delta) = \mathbf{M}(\delta)$ and $\mathbf{F}' = \mathbf{F}$. On the other hand, if (A′) is true, (A) holds with $\mathbf{M}(\delta) = \mathbf{M}'(2\delta)$ and $\mathbf{F}' = \mathbf{F}$.

It is also easily seen that it is sufficient to prove the theorem on the hypothesis that the limit $\lim_{\mathfrak{F}} L_i(X)$ is zero for each X. For, if it is not zero, we can consider instead of $\mathfrak{F}$ the filter $\mathfrak{F}\times\mathfrak{F}$ in the set $\mathbf{I}\times\mathbf{I}$ of pairs (i, i') of elements of $\mathbf{I}$; the sets of $\mathfrak{F}\times\mathfrak{F}$ are the sets $\mathbf{F}\times\mathbf{F}'$ consisting of all pairs (i, i'), where i is in $\mathbf{F}$ and i' in $\mathbf{F}'$. This new filter will also have a countable base; and, since for any X $L_i(X)$ is convergent, we can for any positive δ find $\mathbf{F} \in \mathfrak{F}$ so that $|L_i(X) - L_j(X)| < \delta$ if i and j are in $\mathbf{F}$, i.e. if (i,j) is in $\mathbf{F}\times\mathbf{F}$. Thus if $L(i,j; X) = L(i,X) - L(j,X)$, $L(i,j;X)$ tends to 0 for all X according to the filter $\mathfrak{F}\times\mathfrak{F}$. If then the theorem is proved for the case where the limit is zero, we shall have that for each $\delta > 0$ there exists a set of $\mathfrak{F}\times\mathfrak{F}$, say, $\mathbf{F}\times\mathbf{F}$ (every set of $\mathfrak{F}\times\mathfrak{F}$ contains a set $\mathbf{F}\times\mathbf{F}$), and $\mathbf{M}'$ so that

$$|L_i(X) - L_j(X)| < \tfrac{1}{2}\delta \quad \text{if} \quad (i,j) \in \mathbf{F}\times\mathbf{F} \quad \text{and} \quad X \in \mathbf{M}'.$$

If now i is a fixed element of $\mathbf{F}$, there exists a set $\mathbf{M}'' \in \mathfrak{M}$ such that $|L_i(X)| < \frac{1}{2}\delta$ if $X \in \mathbf{M}''$. It then follows that for all j in $\mathbf{F}$ and all X in $\mathbf{M}' \cap \mathbf{M}''$, $|L_j(X)| < \delta$.

Suppose then that $\lim_{\mathfrak{F}} L_i(X) = 0$, and suppose that (A′) is false, in order to deduce a contradiction. There is then for some $\delta > 0$ an X in every $\mathbf{M}$ and an i in every $\mathbf{F}$ such that

$$|L_i(X)| > \delta. \tag{1}$$

I shall now show that we can choose a sequence of disjoint sets X_m

and a sequence of indices i_m so that each $\mathbf{F}$ in $\mathfrak{F}$ contains an i_m and so that for all m

$$|L_{i_m}(X_m)| > \delta. \tag{2}$$

Take Y_1 to be any set for which (2) holds for some index i_1. A sequence of sets $\{Y_m\}$ will now be defined by induction. Suppose that sets $Y_1,\ldots,Y_m$ and indices $i_1,\ldots, i_m$ have been chosen such that, if

$$Z^m_{(p)} = Y_p - Y_p \cap \Big\{\bigcup_{p<j<m} Y_j\Big\} \quad (p = 1,\ldots, m), \tag{3}$$

then
$$|L(i_p; Z_p^{(m)})| > \delta \quad (p = 1,\ldots, m): \tag{4}$$

say, for definiteness,

$$|L(i_p; Z_p^{(m)})| > \delta + \epsilon_m.$$

Then for each p and m there is a set $\mathbf{M}(i_p; \epsilon_m)$ in $\mathfrak{M}$ such that, if $X \in \mathbf{M}(i_p; \epsilon_m)$, $|L(i_p; X)| < \epsilon_m$. We can choose Y_{m+1} in $\bigcap_{p=1}^{m} \mathbf{M}(i_p; \epsilon_m)$ so that, for some i_{m+1}, $|L(i_{m+1}; Y_{m+1})| > \delta$. There exists by hypothesis a sequence of sets $\mathbf{B}_n$ which form a base of the filter $\mathfrak{F}$; and we can choose Y_{m+1} and i_{m+1} so that i_{m+1} lies in $\mathbf{B}_{m+1}$. Now, if we put

$$Z_p^{(m+1)} = Y_p - Y_p \cap \Big\{\bigcup_{p<j<m+1} Y_j\Big\} \quad \{p = 1,\ldots, (m+1)\},$$

(4) will be satisfied with $(m+1)$ replacing m.

Finally, take
$$X_m = Y_m - Y_m \cap \Big\{\bigcup_{j<m} Y_j\Big\}.$$

Then the $\{X_m\}$ are a sequence of disjoint sets which satisfy (2) with the indices i_m here chosen.

I now proceed to establish the contradiction by constructing from the X_m a set X for which $\lim L_i(X) \neq 0$. We may suppose that $\lim L_i(X_m) = 0$ for each m; otherwise we should already have a contradiction with the hypotheses. We choose a sub-sequence of the integers $\{m_n\}$ as follows. Take $m_1 = 1$, and suppose $m_1,\ldots, m_n$ to have been chosen. Since the L are completely additive, the series $\sum_m L_i(X_m)$ are convergent for all i. There is therefore a number m_n' such that

$$\Big|\sum_{p>m_n'} L(i_{m_r}; X_p)\Big| < \tfrac{1}{6}\delta \quad (r = 1,\ldots, n),$$

and a number m_n'' such that, if $p > m_n''$ and $Z_n = \bigcup_{s=1}^{n} X_{m_s}$,

$$|L(i_p; Z_n)| = \Big|\sum_{r=1}^{n} L(i_p; X_{m_r})\Big| < \tfrac{1}{6}\delta;$$

for, by hypothesis, $\lim_{p\to\infty} L(i_p; Z_n) = 0$. Now choose m_{n+1} to be greater than the larger of m'_n, m''_n, and by the previous construction we have

$$|L(i_{m_{n+1}}; X_{m_{n+1}})| > \delta.$$

Now put $X = \bigcup_n X_{m_n}$. Then

$$\begin{aligned} |L(i_{m_n}; X)| &= \left|\sum_{r=1}^{\infty} L(i_{m_n}; X_{m_r})\right| \\ &\geqslant |L(i_{m_n}; X_{m_n})| - \left|\sum_{r=1}^{m-1} L(i_{m_n}; X_{m_r})\right| - \left|\sum_{r=m+1}^{\infty} L(i_{m_n}; X_{m_r})\right| \\ &\geqslant \delta - \tfrac{1}{6}\delta - \tfrac{1}{6}\delta = \tfrac{2}{3}\delta. \end{aligned}$$

Since there is an i_{m_n} in every set of $\mathfrak{F}$, it follows that $L_i(X)$ does not tend to zero according to the filter $\mathfrak{F}$ and so we have the contradiction.

I make some comments on the theorem.

Note (1). Although it is a restriction on the family $L_i(X)$ to require that all its members be absolutely continuous with respect to a particular mesh, there is always a mesh (and, in fact, an infinite number) with respect to which all the members of any given family are absolutely continuous. This is important in applications.

Given a family $\{L_i(X)\}$ we can construct such a mesh $\mathfrak{M} = \mathfrak{M}\{L_i\}$ as follows. We take as base for the filter all sets $\mathbf{M}(\mathbf{J};\delta)$, where δ runs through all positive numbers, and $\mathbf{J}$ through all *finite* sub-sets $\mathbf{J} = \{i_1,..., i_k\}$, say, of $\mathbf{I}$, and $\mathbf{M}(\mathbf{J};\delta)$ is the set of all X for which

$$|L_i|(X) < \delta \quad \text{if} \quad i \in \mathbf{J}.$$

It is clear that $\mathbf{M}(\mathbf{J};\delta) \cap \mathbf{M}(\mathbf{J}';\delta') = \mathbf{M}(\mathbf{J} \cup \mathbf{J}';\delta'')$, where $\delta'' = \min(\delta, \delta')$, so that the intersection of any two sets of $\mathfrak{M}$ is in $\mathfrak{M}$. Also no set of $\mathfrak{M}$ is empty; for, if $\{X_m\}$ is any sequence of disjoint elements of $\mathbf{K}$, $\sum_{m=0}^{\infty} |L_i|(X_m)$ is convergent for all i, and hence there is an N such that $\sum_{m>N} |L_i|(X_m) < \delta$ for all i in a finite set $\mathbf{J}$. The union of all the X_m for $m > N$ then belongs to $\mathbf{M}(\mathbf{J}, \delta)$.

We may add that the filter $\mathfrak{M}\{L_i\}$ is contained in any mesh $\mathfrak{S}$ with respect to which all the members of the set L_i are absolutely continuous. For, for any i and $\delta > 0$, there is a set $\mathbf{S}(i, \delta)$ in $\mathfrak{S}$ such that $X \in \mathbf{S}(i, \delta)$ implies $|L_i|(X) < \delta$. If X is in the set

$$\mathbf{S}(\mathbf{J}, \delta) = \bigcap_{i \in \mathbf{J}} S(i, \delta),$$

where $\mathbf{J}$ is any finite set of the i, $|L_i|(X) < \delta$ for all i in $\mathbf{J}$; hence $\mathbf{S}(\mathbf{J}, \delta) \subset \mathbf{M}(\mathbf{J}, \delta)$. $\mathfrak{M}\{L_i\}$ consists of all those sub-sets of $\mathbf{K}$ which contain a set $\mathbf{M}(\mathbf{J}, \delta)$, and every set $\mathbf{M}(\mathbf{J}, \delta)$ contains a set of $\mathfrak{S}$ by the argument just given; hence every set of $\mathfrak{M}\{L_i\}$ contains a set of $\mathfrak{S}$, and so, by the condition F 1 for filters, is a set of $\mathfrak{S}$; that is*

$$\mathfrak{M}\{L_i\} \subset \mathfrak{S}.$$

Note (2). The hypothesis that the filter $\mathfrak{F}$ has a countable base is needed because of the countability assumption implicit in complete additivity. I now give a counter-example to show that the theorem breaks down if the filter $\mathfrak{F}$ does not have a countable base—and this even if $\mathfrak{M}$ is measurable.

For all functions $f(t)$ of $L^2(0,1)$ and all sets X Lebesgue measurable in $(0,1)$, take

$$L_f(X) = \int\limits_X f(t)\,dt.$$

Take for filter $\mathfrak{F}$ the filter of weak neighbourhoods of zero in $L^2(0,1)$ and for mesh $\mathfrak{M}$ that defined by Lebesgue measure μ. Each $L_f(X)$ is clearly absolutely continuous with respect to the mesh $\mathfrak{M}$, and as the definition of the filter $\mathfrak{F}$ is that for each fixed $g(x)$ in $L^2(0,1)$

$$\lim_{\mathfrak{F}} \int\limits_0^1 f(x)g(x)\,dx = 0,$$

it is clear that $\lim\limits_{\mathfrak{F}} L_f(X) = 0$ for each X.

Nevertheless, (A) is not satisfied. To see this note that a base of the filter $\mathfrak{F}$ is given by the sets $\mathbf{V}(J, \delta)$, where J is any finite set of elements of $L^2(0,1)$, δ takes all positive values, and

$$\mathbf{V}(J,\delta) = E_f\left[\left|\int\limits_0^1 f(x)\phi(x)\,dx\right| < \delta, \quad \phi \in J\right].$$

Take now any set of $\mathfrak{M}$, say the set M_σ of all X with measure $\mu(X) \leqslant \sigma$, and any set, say $V(J,\delta)$, of $\mathfrak{F}$. Let X be any set of measure σ. By the Riemann-Lebesgue theorem the integral

$$\int\limits_X e^{inx}\psi(x)\,dx$$

tends to zero as n tends to infinity, for any ψ in $L^2(0,1)$; we can therefore find n so that this integral is less than $\delta\sigma$ for each ψ in J; if we then take

$$f(x) = e^{inx}/\sigma$$

* By means of this construction a proof of Theorem 1 based on the 'basic category' theorem can be derived.

we have $f \in V(J,\delta)$, $X \in M_\sigma$, but

$$|L_f|(X) = \int_X |f(x)|\,dx = 1.$$

THEOREM 2. *If a family of c.a.s.f. $\{L_i\}$ defined for all elements of a complete Boolean ring* **K** *converges for each X in* **K** *to a limit following a filter $\mathfrak{F}$ in the set of indices* **I**, *and if $\mathfrak{F}$ has a countable base, then the limit is a c.a.s.f. on* **K**.

It is easy to see that the limit $L(X) = \lim_{\mathfrak{F}} L_i(X)$ is finitely additive. To show that it is completely additive, we shall consider the mesh $\mathfrak{M}\{L_i\}$ defined in Note 1 to Theorem 1. It follows from Theorem 1 that for any $\delta > 0$ there is a set, say $\mathbf{M}(\mathbf{J},\eta)$, in $\mathfrak{M}\{L_i\}$ and F in $\mathfrak{F}$ such that $|L_i|(X) < \delta$ if $i \in F$ and $X \in \mathbf{M}(\mathbf{J},\eta)$.

Now let $\{X_n\}$ be any sequence of disjoint elements of **K**, and put

$$X = \bigcup_{n>0} X_n, \qquad R_m = \bigcup_{n>m} X_n.$$

Then X is the union of the disjoint elements $X_1,\ldots,X_n$ and R_n so that, by finite additivity,

$$L(X) = \sum_1^n L(X_n) + L(R_n).$$

It is proved in Note 1 that, for some n, R_n lies in $\mathbf{M}(\mathbf{J},\eta)$. It follows that for all $i \in F$, $|L_i|(R_n) < \delta$. We can also find F' such that, if i is in F',

$$|L(R_n) - L_i(R_n)| < \epsilon,$$

and taking i in $F \cap F'$ and ϵ arbitrarily small, it follows that $|L(R_n)| \leqslant \delta$. Hence

$$\left|L(X) - \sum_1^n L(X_n)\right| < \delta,$$

and, since δ can be taken arbitrarily small, complete additivity of L follows.*

COROLLARY. *Under the hypotheses of Theorem* 1, *the limit*

$$L(X) = \lim_{\mathfrak{F}} L_i(X)$$

is an absolutely continuous c.a.s.f. with respect to the mesh $\mathfrak{M}$.

Complete additivity follows from Theorem 2, and absolute continuity with respect to the mesh $\mathfrak{M}$ follows by the arguments used in the proof of Theorem 2 to show $|L(R_n)| < \delta$.

* For convergent sequences of c.a.s.f. on a complete family of sets (i.e. with complements) this theorem is proved by Nikodym (**6**).

Like Theorem 1, Theorem 2 breaks down if the hypothesis that the filter $\mathfrak{F}$ has a countable basis is removed; although the assumption could be weakened to the requirement that there is a sequence i_m such that each set of $\mathfrak{F}$ contains a member of i_m: this last assumption is satisfied by the weak neighbourhoods of a point in Hilbert space although they do not have a countable base.

To construct a counter-example, let $F(X)$ be a finitely additive but not countably additive set-function defined for all the sets of a completely additive family $\mathbf{K}$ of sub-sets of a set E.* Now let $\mathfrak{P}$ denote the set of all partitions P of E into a finite number of elements of $\mathbf{K}$. For P, P' in $\mathfrak{P}$, write $P \supset P'$ if every set in P' is the union of sets in P. $\mathfrak{P}$ is thus partially ordered.

For any partition P there exists an infinite number of c.a.s.f. on $\mathbf{K}$ which are equal to $F(X)$ for each of the finite number of sets occurring in the partition P. Let each such function be denoted by $L(i_P; X)$, where i_P runs through a set of indices $\mathbf{I}(P)$. Write $\mathbf{I}$ for the union of all the sets $\mathbf{I}(P)$.

In the set of indices $\mathbf{I}$, consider the filter $\mathfrak{F}$ which has for base the sets $\mathbf{F}(P)$, where, for any P, $\mathbf{F}(P)$ is the union of all the sets $\mathbf{I}(P')$ with $P' \supset P$. Obviously, $\lim_{\mathfrak{F}} L(i; X) = F(X)$ for each element X of $\mathbf{K}$; for we can find a P in $\mathfrak{P}$ which includes X as one of its sets, and every $P' \supset P$ includes a finite number of sets whose sum is X, and so $L(i; X) = F(X)$ if $i \in \mathbf{F}(P)$.

4. Particular cases of Theorem 1

In the literature there exist a number of theorems which are deducible from Theorem 1 by making special assumptions about the filter $\mathfrak{F}$ or the mesh $\mathfrak{M}$. In almost all the cases the set $\mathbf{I}$ has been the set of integers, and the filter $\mathfrak{F}$ has been that consisting of sets of integers whose complements are finite: that is to say, the theorems have been concerned with sequences of c.a.s.f. These cases, and others, are included in the following theorem.

THEOREM 3. *Under the hypotheses of Theorem* 1, *if the convergence of the c.a.s.f. to zero in the mesh* $\mathfrak{M}$ *is uniform for all* i *in the complement of any set* $\mathbf{F}$ *of the filter* $\mathfrak{F}$, *then the convergence of the c.a.s.f. to zero is uniform for all* i *throughout the set* $\mathbf{I}$.

* The proof of existence of such a set-function seems to require the axiom of choice and the well-ordering theorem.

The hypothesis of this theorem is that for each $\mathbf{F}$ of $\mathfrak{F}$ and $\delta > 0$ there is a set $\mathbf{M}(\mathbf{F};\delta) \in \mathfrak{M}$ such that $|L_i|(X) < \delta$ if $i \in \mathfrak{C}\mathbf{F}$ and $X \in \mathbf{M}(\mathbf{F};\delta)$.

Now, from Theorem 1, for any $\delta > 0$ there is an $\mathbf{F} \in \mathfrak{F}$ and an $\mathbf{M} \in \mathfrak{M}$ such that $|L_i|(X) < \delta$ if $i \in \mathbf{F}$ and $X \in \mathbf{M}$.

If, then, $X \in \mathbf{M} \cap \mathbf{M}(\mathbf{F};\delta)$, $|L_i|(X) < \delta$ for all i, and so the convergence to zero is uniform throughout $\mathbf{I}$.

If a set of i is finite, convergence of $L_i(X)$ is certainly uniform throughout that set; hence we get the following theorem for sequences:

THEOREM 3*a*. *If $\{L_m(X)\}$ is a sequence of c.a.s.f. defined on a complete Boolean ring, convergent for each X as $m \to \infty$, and absolutely continuous in a mesh $\mathfrak{M}$ for each m, then the c.a.s.f. are uniformly absolutely continuous in the mesh $\mathfrak{M}$, and the limit function is absolutely continuous in the mesh $\mathfrak{M}$.*

If the mesh $\mathfrak{M}$ is derived from a measure μ defined on a completely additive family of sets, this theorem includes theorems of Lebesgue (**7**), Vitali (**8**), and Hahn (**1**).

Another special case is that in which the elements of $\mathbf{K}$ are the set of all the sub-sets of a countable set (say the set of integers) and the mesh $\mathfrak{M}$ has as base the sets $\mathbf{M}(N)$, where N runs through all the integers, and $\mathbf{M}(N)$ consists of all sets of integers whose lower bound is greater than N. In this case we get another theorem of Hahn (**1**). The c.a.s.f. here are represented by absolutely convergent series:

$$L_i(X) = \sum_{m \in X} l_{im}, \quad \text{where} \quad \sum_m |l_{im}| < \infty.$$

The theorems become:

THEOREM 3*b*. *If a family of absolutely convergent series $L_i = (l_{im})$ is such that for some filter $\mathfrak{F}$ with a countable base in the set of indices* $\mathbf{I}$

$$\lim_{\mathfrak{F}} \sum_{m \in X} l_{im}$$

exists for every set of integers X, then for every $\delta > 0$ there is a set $F(\delta) \in \mathfrak{F}$ and a number $N(\delta)$ such that for all i in $F(\delta)$

$$\sum_{m > N(\delta)} |l_{im}| < \delta.$$

In particular, if the family is a sequence, it is uniformly convergent.

From Theorem 3 certain results of Dubrovsky (**13**) can be deduced. Thus his theorem that, if $\{L_i\}$ is a convergent sequence of c.a.s.f

and $\{X_n\}$ is any sequence of disjoint elements, then $\sum_{p>n} L_i(X_p)$ converges to zero as $n \to \infty$, uniformly in i, follows from Theorem 3*b*, since $L_i(X_n) = l_{in}$ is a sequence of absolutely convergent series.

Other theorems arise on making special hypotheses concerning the mesh $\mathfrak{M}$.

THEOREM 4. *If the hypotheses of Theorem* 1 *hold, and if each set of the mesh* $\mathfrak{M}$ *contains an element* Z *of* $\mathbf{K}$ *such that convergence of* $L_i(X)$ *according to the filter* $\mathfrak{F}$ *is uniform for all elements* X *disjoint to* Z, *then convergence is uniform for all elements of* $\mathbf{K}$.

Moreover, $|L_i - L_j|(X)$ *tends to zero according to the filter* $\mathfrak{F} \times \mathfrak{F}$, *and, if* $L(X)$ *is the limit of* $L_i(X)$, $|L - L_i|(X) \to 0$.

By Theorem 1 there exists, for any $\delta > 0$, $\mathbf{F}(\delta)$ in $\mathfrak{F}$ and $\mathbf{M}(\delta)$ in $\mathfrak{M}$ such that, if $i \in \mathbf{F}(\delta)$ and $X \in \mathbf{M}(\delta)$, then $|L_i|(X) < \delta$; and we may suppose $\mathbf{M}(\delta)$ to satisfy the condition M1, that any element contained in an element of $\mathbf{M}(\delta)$ is in $\mathbf{M}(\delta)$. Let $Z \in \mathbf{M}(\delta)$ be such that convergence is uniform over all elements disjoint to Z: then for any $\delta > 0$ there is $\mathbf{F}(Z;\delta)$ in $\mathfrak{F}$ such that, if i, j are in $\mathbf{F}(Z;\delta)$ and $X \cap Z$ is null, then $|L_i(X) - L_j(X)| < \delta$. Now, if Y is any element of $\mathbf{K}$, $Y = (Y \cap Z) \cup (Y - Y \cap Z) = Y_1 \cup Y_2$ say, where Y_1 and Y_2 are disjoint elements, and Y_2 is disjoint to Z. Thus

$$|L_i(Y) - L_j(Y)| = |L_i(Y_1) - L_j(Y_1)| + |L_i(Y_2) - L_j(Y_2)|.$$

The first term on the right-hand side is less than δ if i and j are in $\mathbf{F}(\delta)$, since Y_1 is in $\mathbf{M}(\delta)$, and the second term is less than δ if i and j are in $\mathbf{F}(Z, \delta)$. Hence the left-hand side is less than δ if i and j are in $\mathbf{F}(\delta) \cap \mathbf{F}(Z;\delta)$, and the uniform convergence follows.

For the second part, $|L_i - L_j|(Y)$ is the upper bound of

$$|L_i(X) - L_j(X)|$$

for all $X \subset Y$, and, since $L_i(X) - L_j(X)$ tends to 0 uniformly, it follows that $|L_i - L_j|(Y)$ tends to 0. The same argument proves the last part.

In the case of families of Lebesgue integrals on Lebesgue measurable sets in a set of finite measure, the uniform convergence required by the theorem follows from Egoroff's theorem in the case of integrals of a convergent sequence of functions, and we get some well-known theorems of Lebesgue.

In the case of families of absolutely convergent series, discussed under Theorem 3*b*, each set $\mathbf{M}$ of $\mathfrak{M}$ contains sets of integers whose

complements are finite; obviously convergence of $L_i(X)$ over sets in these complements is uniform. We thus have:

THEOREM 4 *a*. *If a family $L_i = (l_{im})$ of absolutely convergent series is such that for every set of integers, X,*

$$\lim_{\mathfrak{F}} \sum_{m \in X} l_{im}$$

exists according to a filter $\mathfrak{F}$ with a countable base in the set of indices $\mathbf{I}$, *then the family is strongly convergent, i.e.*

$$\sum | l_{im} - l_{jm} | \to 0.$$

This generalizes a theorem of Banach [(**2**) 137] that weakly convergent sequences in the space of absolutely convergent series converge strongly in that space.

As a counter-example for Theorem 4, consider the set functions

$$L_t(X) = \int_X e^{itx}\, dx,$$

defined for all measurable sets X in $(0, 1)$ and all real t. As $t \to \infty$, $L_t(X) \to 0$; but $|L_t|(X) = \mu(X)$ for all t. We can conclude that there is no set X of arbitrarily small measure in whose complement $L_t(Y) \to 0$ uniformly.

5. An application to the theory of summation of series

A matrix $(a_{mn}) = A$ is said to sum a sequence $\{y_n\}$ to the limit z if $z = \lim\limits_{m \to \infty} \sum\limits_{n=0}^{\infty} a_{mn} y_n$. A theorem of Steinhaus [see (**9**) 392, VI] states that any matrix which sums all convergent sequences to their actual limits cannot sum all bounded sequences. This is contained in the following:

THEOREM 5. *A matrix which sums all convergent sequences of* 0's *and* 1's *to their actual limits, cannot sum all sequences of* 0's *and* 1's *to a limit.*

For any set X of integers, put

$$A_m(X) = \sum_{n \in X} a_{mn}.$$

If A sums all sequences of 0's and 1's, then $\lim A_m(X)$ as $m \to \infty$ must exist for every X. It then follows from Theorem 1 that for every $\delta > 0$ there exist M and N such that for $m \geqslant M$

$$\sum_{n > N} |a_{mn}| < \delta.$$

Now let
$$c_n = \begin{cases} 0 & (n < N), \\ 1 & (n \geqslant N). \end{cases}$$
Then, for $m \geqslant M$,
$$\left|\sum_{n=0}^{\infty} a_{mn} c_n\right| \leqslant \sum_{N}^{\infty} |a_{mn}| < \delta,$$
so that A sums the sequence $\{c_n\}$ to a limit less than 1, whereas its correct limit is 1.

6. Completely additive set functions with values in linear topological spaces

An interesting extension of Theorem 1 is to the case where the functions $L(X)$ have as their values not real (or complex) numbers but elements in a linear topological space [see (**10**), (**11**)]. By this we mean a linear T_1 space in which addition of elements and multiplication of elements by real numbers are continuous in the topology.* The extension of the definition of a c.a.s.f. is as follows:

$L(X)$, defined for each element X in a complete Boolean ring **K**, and having for each X in **K** as value an element in a linear topological space, is said to be a 'c.a.s.f. on **K**' if, for every sequence of disjoint elements X_n in **K** whose union is X, the sum $\sum L(X_n)$ exists as an element in the space, and is equal to $L(X)$.

Further, we say that $L(X)$ is *absolutely continuous with respect to a mesh* $\mathfrak{M}$ if, for each neighbourhood of zero U of the space, there is an **M** in $\mathfrak{M}$ such that $L(X)$ is in U for all X in **M**.

By arguments very similar to those used in the proof of Theorem 1, we get:

THEOREM 1 *a*. *Let $L_i(X)$ be a family of c.a.s.f. defined for all elements X of a complete Boolean ring* **K**, *taking values in a linear topological space R, and, for each i, absolutely continuous with respect to a mesh $\mathfrak{M}$ on* **K**; *for each X in* **K** *let $L_i(X)$ tend to a limit according to a filter with a countable base on the set of indices* **I**: *then, for any neighbourhood of zero U of R, there is a set* **M** *in $\mathfrak{M}$ and a set F in $\mathfrak{F}$ such that $L_i(X)$ is in U if X is in M and i is in F.*

This theorem holds, in particular, for Banach spaces.

* The space is thus required to satisfy the conditions of (**10**). The axioms for the neighbourhoods of zero in (**11**) are equivalent to these conditions, save that the axiom that there exists a sequence of neighbourhoods of zero whose intersection is the zero element can be omitted, and replaced by the assumption that the intersection of all the neighbourhoods of zero is the zero element. Completeness need not be assumed.

The theorem has an application in connexion with theorems of Hobson [(**12**) Chap. VII], which are generalizations of the theorem of Lebesgue cited above, and which give necessary and sufficient conditions that sequences of integrals of the form

$$\int_a^b k_n(x,t)f(t)\,dt \tag{1}$$

shall converge uniformly, or in mean, etc., for certain classes of functions $f(t)$. In most cases the proofs of the necessity of the conditions are more difficult than those of their sufficiency: the Theorem 1 *a* provides the proof of necessity in the case where the class of functions $f(t)$ considered is the class of all bounded measurable functions. For this class contains as a sub-set the class of all characteristic functions of sets; and we can deduce from Theorem 1 *a* that a necessary condition that the integral (1) should tend to a limit uniformly (or everywhere, or in mean p, etc.) is that

$$\phi_n(X,x) = \int_X k_n(x,t)\,dt$$

tends to zero uniformly in n as the measure of the set X tends to zero: where we interpret '$\phi_n(x)$ tends to zero' to mean 'tends uniformly' (or 'everywhere', or 'in mean p', etc.) according to the fashion in which the integral (1) is required to tend to zero.

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DIFFERENTIAL EQUATIONS WITH POLYNOMIAL SOLUTIONS

By W. W. SAWYER (*Achimota*)

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1. The elementary methods used in mathematical physics for finding the 'eigenvalues' and 'eigenfunctions' of differential equations mostly depend upon the eigenfunctions being polynomials.* Usually—as, for example, with the Legendre, Hermitian, and Laguerre polynomials—the differential equation is of hypergeometric type. It does not ever seem to have been established whether any differential equation other than the hypergeometric can have all its eigenfunctions polynomials. The present paper establishes necessary and sufficient conditions that the differential equation

$$a(x)y''+b(x)y'+c(x)y = \lambda\{A(x)y''+B(x)y'+C(x)y\} \tag{1}$$

shall have a polynomial solution of exact degree n corresponding to the parameter value λ_n for all non-negative integral n.

The procedure is to form a sequence of differential equations, such that the nth equation has for its general solution $d^n y/dx^n$, where y is the general solution of the original equation (1). If, for $\lambda = \lambda_n$, the original equation has a polynomial of exact degree n as a particular solution, the nth equation of the sequence will have a non-zero constant as a particular solution.

In particular, for $\lambda = \lambda_0$, the original equation is satisfied by $y =$ constant, and hence

$$c(x) = \lambda_0 C(x). \tag{2}$$

Dividing equation (1) by equation (2) we obtain

$$\lambda_0(a_0 y''+b_0 y'+y) = \lambda(A_0 y''+B_0 y'+y), \tag{3}$$

where $a_0 = a(x)/c(x)$, and so on.

On differentiating equation (3) we obtain the differential equation for $y_1 = dy/dx$,

$$\lambda_0\{a_0 y_1''+(a_0'+b_0)y_1'+(b_0'+1)y_1\} = \lambda\{A_0 y_1''+(A_0'+B_0)y_1'+(B_0'+1)y_1\}. \tag{4}$$

* See, for example, Ruark and Urey, *Atoms, Molecules and Quanta*, Appendixes IV and V.

Since $y_1 =$ constant is to satisfy equation (4) for $\lambda = \lambda_1$, it follows that

$$\lambda_0(b_0'+1) = \lambda_1(B_0'+1). \tag{5}$$

Dividing equation (4) by equation (5) we have

$$\lambda_1(a_1 y_1''+b_1 y_1'+y_1) = \lambda(A_1 y_1''+B_1 y_1'+y_1), \tag{6}$$

where $$a_1 = a_0/(b_0'+1), \qquad b_1 = (a_0'+b_0)/(b_0'+1) \tag{7}$$

with similar equations for A_1 and B_1.

Since equation (6) has the same form as equation (3), the process may in general be repeated indefinitely. An exception will arise if, for any n, $b_n'+1$ or $B_n'+1$ is zero, or if in equation (1) either $c(x)$ or $C(x)$ is zero. This point will be discussed later. In general we shall obtain a series of differential equations

$$\lambda_n(a_n y_n''+b_n y_n'+y_n) = \lambda(A_n y_n''+B_n y_n'+y_n), \tag{8}$$

defined by the relations

$$a_{n+1} = a_n/(b_n'+1), \qquad b_{n+1} = (a_n'+b_n)/(b_n'+1), \tag{9}$$

and the corresponding equations for A_n and B_n. At each step a condition arises, similar to (2) and (5), namely,

$$\lambda_n(b_n'+1) = \lambda_{n+1}(B_n'+1). \tag{10}$$

By equations (9), a_n/a_{n+1} may be substituted for $b_n'+1$ and a similar substitution made on the right-hand side of the equation. The resulting equation shows that $\lambda_n a_n/A_n$ is independent of the value of n: that is,

$$\lambda_n a_n/A_n = \lambda_0 a_0/A_0 = \psi(x), \text{ say.} \tag{11}$$

The equation (8) may be abbreviated to

$$\lambda_n e_n y_n = \lambda E_n y_n,$$

where e_n stands for the operator $a_n D^2+b_n D+1$, and E_n represents the corresponding operator in A_n, B_n.

All steps of the argument are reversible. If equation (11) holds for the sequence of functions defined by equation (9), equation (1) will have a polynomial solution of exact degree n for all non-negative n. For condition (10) can be deduced from (11) and (9), so that $b_n'+1$ will be an integrating factor for both the right-hand and left-hand sides of the $(n+1)$th differential equation. If df/dx is a solution of the differential equation for y_{n+1}, it does not follow that f is a solution of the equation for y_n, owing to the appearance of a constant of integration, but rather that

$$\lambda_n e_n f = \lambda E_n f+K, \tag{12}$$

where K is independent of x but may depend on the parameter λ. Since, however,

$$\lambda_n e_n 1 = \lambda E_n 1 + \lambda_n - \lambda,$$

it is clear that $(\lambda-\lambda_n)f+K$ is a solution of the equation for y_n. In virtue of (12), this solution may also be written in the form

$$\lambda_n(e_n-1)f-\lambda(E_n-1)f:$$

that is, $$\lambda_n(a_n f''+b_n f')-\lambda(A_n f''+B_n f'),$$

which depends only on f' and its differential coefficient.

By successive application of this formula it is possible, by starting from the fact that $y_n = 1$ satisfies the nth equation for $\lambda = \lambda_n$, to obtain the polynomial of the nth degree that satisfies the original equation.

2. It is possible to give a more convenient form to condition (11). Let $y = u$ and $y = v$ be two independent solutions of equation (1) for $\lambda = 0$, and $y = U$ and $y = V$ be two independent solutions of equation (1) for $\lambda = \infty$. The nth derivatives, u_n, v_n, of u and v, will satisfy equation (8) for $\lambda = 0$, and in the general case at present being considered will be independent solutions. For $\lambda = 0$, equation (8) reduces to $e_n y_n = 0$, which must accordingly be the same equation as

$$\begin{vmatrix} y_n'' & y_n' & y_n \\ u_n'' & u_n' & u_n \\ v_n'' & v_n' & v_n \end{vmatrix} = 0,$$

i.e. $q_n y_n''-q_n' y_n'+q_{n+1} y_n = 0$, where $q_n = u_{n+1} v_n - v_{n+1} u_n$.

By comparing coefficients, $a_n = q_n/q_{n+1}$. Similarly and with a similar notation, $A_n = Q_n/Q_{n+1}$. On substituting in (11) it follows that

$$Q_{n+1}/q_{n+1} = (Q_n/q_n)\psi(x)/\lambda_n,$$

whence $$Q_n/q_n = [\psi(x)]^n \chi(x)\phi(n), \tag{13}$$

where

$$\chi(x) = Q_0/q_0, \qquad \phi(n) = 1/\lambda_{n-1}\lambda_{n-2}...\lambda_0, \qquad \phi(0) = 1.$$

Conditions (13) can be written more symmetrically. Writing

$$\lambda_n = s_n/S_n, \qquad \psi(x) = f(x)/F(x), \qquad G(n) = S_{n-1}S_{n-2}...S_0,$$

$$G(0) = 1, \qquad g(n) = s_{n-1}s_{n-2}...s_0, \qquad g(0) = 1,$$

$$\chi(x) = H(x)/h(x),$$

we can see that equations (13) will hold if and only if, for some $\Phi_n(x)$,

$$\left.\begin{aligned} Q_n(x) &= [F(x)]^{-n}H(x)\Phi_n(x)G(n) \\ q_n(x) &= [f(x)]^{-n}h(x)\Phi_n(x)g(n) \end{aligned}\right\}. \tag{14}$$

From any two sequences Q_n, q_n having the form of the above expressions—the function $\Phi_n(x)$ being, of course, the same in the two expressions—it is possible to deduce a differential equation of the desired type.

By 'a sequence q_n' we understand a set of functions

$$q_n = u_{n+1}v_n - v_{n+1}u_n,$$

formed from the derivatives of any pair of functions of x, u and v. A set of expressions q_n will form such a sequence if and only if they satisfy the relation

$$q_n q_{n+2} - q_{n+1}^2 = q'_n q'_{n+1} - q_{n+1} q''_n.$$

This relation may be written

$$\Delta\left(\frac{q_{n+1}}{q_n}\right) + \frac{q_{n+1}}{q_n}\frac{d}{dx}\left(\frac{q'_n}{q_{n+1}}\right) = 0. \tag{15}$$

The numbers s_n, S_n, from which $g(n)$ and $G(n)$ are built up, should be taken at most as quadratic expressions in n. For, when $\lambda = \lambda_n$, equation (1) has a solution for which y is asymptotic to x^n for large x. By considering the highest powers of x that occur when this solution is substituted in equation (1), it may be seen that λ_n is the quotient of two functions of n, each a polynomial of degree two at most. Without loss of generality s_n and S_n may be taken to be equal to the numerator and denominator respectively of λ_n. No restrictions need be placed on $f(x)$, $F(x)$, $H(x)$, and $h(x)$.

The problem is thus equivalent to the following: *to find the most general $\Phi_n(x)$ such that two distinct sequences Q_n, q_n exist, each obeying an equation of the form* (15), *and related to $\Phi_n(x)$ by equations of type* (14), *in which the functions $G(n)$, $g(n)$ are restricted as mentioned above.*

It can easily be shown that, if $\Phi_n(x)$ is a rational function of n, x being regarded as a parameter, then $F(x)$ and $f(x)$ will be quadratic functions of x and the resulting differential equation will be hypergeometric. This result can be proved by substituting in equation (15) the expression for q_n that appears in (14), and considering the behaviour for large n. It can also be proved by an argument in

which calculation plays a much smaller part,* if the differential equation is assumed to be Fuchsian with polynomial coefficients.

Differential equations of the type sought, other than hypergeometric equations, can therefore exist only if the conditions stated above can be satisfied by a function $\Phi_n(x)$ which is not a rational function of n. (It is assumed that $\Phi_n(x)$ does not contain any factor of the form $\{F(x)\}^n$.)

3. It is possible to interpret the meaning of the functions $F(x)$, $f(x)$, and $\Phi_n(x)$ in terms of the singularities of the differential equation (1). This interpretation is not used anywhere in this paper as a foundation for logical deduction, and it will therefore only be sketched in general outline, without going into details of any exceptional cases.

Suppose then that the functions u, v, U, V have only regular singularities. In the neighbourhood of any singularity a, u and v will behave like $(x-a)^\alpha$ and $(x-a)^\beta$, so that $q_0 = u'v-uv'$ will behave like $(x-a)^{\alpha+\beta-1}$. It is well known† that q_0 must be of the form

$$P_0(x) \prod (x-a)^{\alpha+\beta-1},$$

where $P_0(x)$ is a polynomial. The points of which a is typical are the singularities; the zeros of $P_0(x)$ are known as apparent singularities or *nebenpunkte*. If $x = k$ is a *nebenpunkt*, the solutions u and v will behave like $(x-k)^\gamma$ and $(x-k)^\delta$, where γ and δ are positive integers, or zero. If k is a simple root of $P_0(x) = 0$, $\gamma = 0$ and $\delta = 2$.

Singularities and *nebenpunkte* affect the sequence q_n in very different ways. If a is a pole or a branch point of u, it will necessarily be a pole or a branch point of the nth derived function u_n, the index α being replaced by $\alpha-n$. Thus, if a is a singularity both of u and v, q_n will contain the factor $(x-a)^{\alpha+\beta-2n-1}$. In equation (14) this suggests that $f(x)$ contains the factor $(x-a)^2$ and $h(x)$ the factor $(x-a)^{\alpha+\beta-1}$. This is necessarily so if we assume $\Phi_n(x)$ not to contain any factors that could be absorbed into $f(x)$ or $h(x)$. This assumption involves no loss of generality.

If a is a singularity of u but not of v—and this is the case of most interest in connexion with our problem—v_n will in general have the index zero for all n. Accordingly q_n will contain the factor $(x-a)^{\alpha-n-1}$:

* By using a result given in W. W. Sawyer, *Quart. J. of Math.* (Oxford), 15 (1944), 34–9.

† Riemann, 1857. Klein, *Hypergeometrische Function* (1894), section I (D).

$f(x)$ will contain $(x-a)$ instead of its square, and $h(x)$ will contain $(x-a)^{\alpha-1}$.

On the other hand, it is impossible for $f(x)$ to contain any factor which is a power of $(x-k)$ where k is a *nebenpunkt*. If $f(x)$ contained a positive power of $(x-k)$, for sufficiently large n, the index of $(x-k)$ in $h(x)[f(x)]^{-n}$ would be negative: that is to say, q_n would have a pole at k. But $q_n = u_{n+1}v_n - v_{n+1}u_n$ cannot have a pole if there is no pole in u and v. Hence $f(x)$ cannot contain a positive power of $(x-k)$.

If $f(x)$ contains a negative power of $(x-k)$, the index of q_n at $x = k$ must increase in arithmetic progression as n takes successive values. But, in general, differentiation decreases the indices of u and v, and hence of q. The only exception is when one of the functions u, v has index zero, since differentiation of

$$\epsilon_1+\epsilon_2(x-k)^{m+1}+\ldots \text{ of index } 0,$$

leads to

$$(m+1)\epsilon_2(x-k)^{m}+\ldots \text{ of index } m.$$

When m is greater than 0, the index of u' exceeds that of u. It is to be noted that, when m exceeds 0, u'' will have an index lower than that of u', so that an increase of index cannot occur in two consecutive differentiations of the same function.

Accordingly, if the index of q_n is to increase steadily with n, it is necessary that the index zero arise alternately in the derivatives of u and v: that is, that u, v', u'', v''',... shall all have index 0 at $x = k$. But, if v' has index 0, v itself must contain the first power of $(x-k)$: thus u, v have indices 0, 1 so that k is not a *nebenpunkt* at all.

One special case needs to be mentioned. If u, v have the forms

$$\epsilon_1+\epsilon_2(x-k)^{m+1}\ldots \quad\text{and}\quad \epsilon_3(x-k)^{m+1}+\epsilon_4(x-k)^{r}+\ldots \quad (r > m+1),$$

both u' and v' will begin with the mth power of $x-k$. Some linear combination of u' and v', however, will begin with the $(r-1)$th power, and the effective indices will be m and $r-1$. Since m must be at least 1, if the equation for u, v is to have a *nebenpunkt*, and since $r-1$ is bigger than m, both indices will decrease in going from u', v' to u'', v''. Thus this exceptional case does not allow the indices to increase steadily. When the index of v is different from $m+1$ the argument of the previous paragraph applies.

The same type of argument shows that it is impossible for the index of $x-k$ to be stationary as n takes all possible values. It is

thus impossible for $x-k$ to be a factor of $h(x)$. Accordingly the factors of q_n which correspond to *nebenpunkte* in the equation $e_n y_n = 0$ can only appear in the function $\Phi_n(x)$. Also, since any factors corresponding to genuine singularities can, as has been shown, be absorbed into the factors $h(x)[f(x)]^{-n}$, $\Phi_n(x)$ may be chosen to consist only of factors corresponding to *nebenpunkte*.

But the same function $\Phi_n(x)$ occurs in the expressions for q_n and Q_n. It follows that the equations $e_n y_n = 0$ and $E_n y_n = 0$ have the same *nebenpunkte*: that is to say, equation (8) has the same *nebenpunkte* for $\lambda = 0$ and $\lambda = \infty$. This means that for all values of λ, the *nebenpunkte* of equation (8) are the roots of $\Phi_n(x) = 0$.

The proof is as follows. On substituting for q_n the value given in equation (14), the left-hand side of equation (8) becomes

$$e_n y_n = \frac{f(x)\Phi_n(x)}{s_n \Phi_{n+1}(x)} y_n'' - \frac{f(x)\Phi_n(x)}{s_n \Phi_{n+1}(x)} \left\{ \frac{h'(x)}{h(x)} - \frac{nf'(x)}{f(x)} + \frac{\Phi_n'(x)}{\Phi_n(x)} \right\} y_n' + y_n,$$

so that $s_n e_n y_n$ is of the form

$$\frac{\omega\Phi_n(x)}{\Phi_{n+1}(x)} y_n'' + y_n' \left\{ \theta \frac{\Phi_n(x)}{\Phi_{n+1}(x)} - \omega \frac{\Phi_n'(x)}{\Phi_{n+1}(x)} \right\} + k y_n \tag{16}$$

and $S_n E_n y_n$ is of the same form.

The differential equation obtained by equating to zero an expression of the form (16) will in general have the roots of $\Phi_n(x) = 0$ not as *nebenpunkte* but as actual singularities. The condition for these points merely to be *nebenpunkte* is that $\Phi_n(x)$ shall be a factor of

$$k\Phi_{n+1}^2(x) + (\theta - \omega')\Phi_{n+1}(x)\Phi_n'(x) + \omega\{\Phi_n'(x)\Phi_{n+1}'(x) - \Phi_{n+1}(x)\Phi_n''(x)\}. \tag{17}$$

This condition may be found in various ways: one method, which is most closely related to the general methods of this paper, is to differentiate the expression (16)—as would be done to form the expression $e_{n+1} y_{n+1}$—and then to use the fact that $e_{n+1} y_{n+1} = 0$ has singularities (genuine or apparent) only where $\omega\Phi_{n+1}(x) = 0$, so that the factor $\Phi_n(x)$, which appears in the coefficient of y_{n+1}'', must be a factor of all the coefficients in the equation.

But, since the expression (17) is a linear homogeneous expression in k, θ, ω, if $\Phi_n(x)$ is a factor of two different expressions of this form containing k_1, θ_1, ω_1 and k_2, θ_2, ω_2 respectively, it will also be a factor of the expression containing the linear combination $k_1 - \lambda k_2$, $\theta_1 - \lambda\theta_2$, $\omega_1 - \lambda\omega_2$.

Accordingly, since the condition is satisfied by the expressions $s_n e_n y_n$ and $S_n E_n y_n$, it will be satisfied by the linear combination of these, $s_n e_n y_n - \lambda S_n E_n y_n$, whatever λ may be. And equation (8), which is formed by equating this differential expression to zero, accordingly has the *nebenpunkte* given by $\Phi_n(x) = 0$, for all λ.

4. Exceptions and apparent exceptions

The arguments used above fail (i) if in equation (1) either c or C is identically zero, (ii) if b'_n+1 or B'_n+1 is identically zero for any n, (iii) if u_n and v_n are linearly dependent, or if U_n and V_n are linearly dependent.

If $c = 0$, $C \neq 0$, equation (1) is satisfied by $y =$ constant for $\lambda = 0$, and so $\lambda_0 = 0$.

If the process of formation of equations (8) can be carried out up to $n = r$, and then $b'_r+1 = 0$, $B'_r+1 \neq 0$, it follows that $\lambda_{r+1} = 0$.

Similarly, the exceptions $C = 0$, $c \neq 0$ and $B'_r+1 = 0$, $b'_r+1 \neq 0$, can arise only if some eigenvalue, λ_0 or λ_{r+1} respectively, is infinite.

If u_n, v_n are linearly dependent, so that $\sigma u_n + \tau v_n = 0$ for constants σ, τ, not both zero, $\sigma u + \tau v$ will be a non-zero polynomial and will satisfy equation (1) for $\lambda = 0$. Thus zero is again an eigenvalue. U_n, V_n linearly dependent corresponds to an infinite eigenvalue.

In general these apparent exceptions can be removed by a change of parameter. The form of equation (1) is unaltered by the projective substitution

$$\lambda = \frac{\epsilon_1 + \epsilon_2 \Lambda}{\epsilon_3 + \epsilon_4 \Lambda},$$

where the coefficients ϵ_1, ϵ_2, ϵ_3, ϵ_4 are constants. The eigenvalues λ_n in general form a discrete set. If the constants ϵ are so chosen that ϵ_1/ϵ_3 and ϵ_2/ϵ_4 do not belong to this set, 0 and infinity will not be among the eigenvalues for the new parameter Λ, and with the new parameter it will be possible to follow the ordinary procedure and argument.

A genuine exception arises if every number is an eigenvalue of λ. This case arises if $c = C = 0$, or if for some r, b'_r+1 and B'_r+1 are both zero. In the first case, $y =$ constant is a solution of (1) for all λ. In the second case, equation (1) will have a polynomial solution of exact degree $r+1$ for all λ. The coefficients of this polynomial will in general be functions of λ.

In the first case y', and in the second case y_{r+2}, satisfies a differential equation of the first order, of the form

$$az'+bz = \lambda(Az'+Bz). \tag{18}$$

Since the original equation (1) has polynomial solutions of every degree, and since (18) is satisfied by y' or y_{r+2}, the same must be true of equation (18). The problem thus reduces to one similar to the original problem, but for differential equations of the first instead of the second order.

It can be shown by the methods used above that the general solution of (18) must be reducible by change of the parameter λ and by a linear substitution for x to one of the standard forms,

$$z = kx(x+\lambda)^{\lambda-1}, \qquad z = kx^{\lambda}, \quad \text{or} \quad z = k(x+\lambda)^{\lambda}. \tag{19}$$

The general solution y of equation (1) belongs to the class of functions obtained by integrating z a finite number of times; y accordingly cannot have more than one finite singularity, and a singularity at $x = \infty$. Both singularities are regular. The exceptional case therefore does not lead to any differential equation of interest.

THE FRACTIONAL DIMENSION OF A SET DEFINED BY DECIMAL PROPERTIES

By H. G. EGGLESTON (*Swansea*)

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WHEN any real number x $(0 \leqslant x \leqslant 1)$ is expressed as a decimal in the scale N (i.e. involving digits 0, 1, 2,..., $N-1$), let $P(x, i, r)$ denote the number of times the digit r occurs amongst the first i digits of this decimal.

The following theorem has been conjectured by I. J. Good.†

THEOREM. *The set S of those x $(0 \leqslant x \leqslant 1)$ for which*

$$\lim_{i\to\infty} \frac{P(x,i,r)}{i} = p_r \quad (r = 0, 1,..., N-1)$$

where $0 \leqslant p_r \leqslant 1$, $\sum_0^{N-1} p_r = 1$, *has fractional dimension* α, *given by*

$$N^{-\alpha} = \prod_{r=0}^{N-1} p_r^{p_r}.$$

[A linear set S has *fractional dimension* α if $\Lambda_\beta(S) = 0$ for $\beta > \alpha$ and $\Lambda_\beta(S) = \infty$ for $\beta < \alpha$, where

$$\Lambda_\beta(S) = \lim_{\delta\to 0}\Big\{\text{lower bound} \sum_{U(S,\delta)} d^\beta\Big\}.$$

$U(S, \delta)$ is any covering of S by intervals of length d less than δ: the sum is over each member of $U(S, \delta)$ and the lower bound is over all possible $U(S, \delta)$.]

The proof of this theorem is given here for the case $p_r \neq 0$ $(r = 0,..., N-1)$. A similar method may be applied when some of the p_r are zero. I am indebted to the referee for elucidating a number of ambiguities and for offering several helpful suggestions.

(*a*) *Dimension of $S \leqslant \alpha$.* Let $\epsilon > 0$ be given, and $R(i, \epsilon)$ be the set of all terminating i-figured decimals x which satisfy

$$0 \leqslant x \leqslant 1; \quad (p_r - \epsilon)i \leqslant P(x, i, r) \leqslant (p_r + \epsilon)i \quad (r = 0, 1,..., N-1). \tag{1}$$

Let $X(i, \epsilon)$ be the set of closed intervals whose left-hand end-points are the points of $R(i, \epsilon)$ and which are all of length $1/N^i$.

For every $\epsilon > 0$,
$$\sum_{i_0=1}^{\infty} \prod_{i=i_0}^{\infty} X(i, \epsilon) \supset S. \tag{2}$$

† *Proc. Cambridge Phil. Soc.* 37 (1941), 200.

Thus, in order to prove that the dimension of $S \leqslant \alpha$, it is sufficient to show that, for every integer i_0 and every $\eta > 0$, there is an $\epsilon > 0$ (which is a function of η only) such that the dimension of $\prod_{i=i_0}^{\infty} X(i,\epsilon)$ is less than $\alpha+\eta$, or equal to it.

Since $\prod_{i=i_0}^{\infty} X(i,\epsilon)$ is covered by the set of intervals $X(i,\epsilon)$ $(i \geqslant i_0)$, and these intervals are of length $1/N^i$, it is sufficient to show that, given $\eta > 0$, there is an $\epsilon > 0$ such that

$$\sum_{X(i,\epsilon)} d^{\alpha+\eta} \to 0 \quad (i \to \infty).$$

Let $\mathfrak{N}(A)$ denote the number of elements in a finite set A. Then it is sufficient to show that there is an $\epsilon > 0$ such that

$$\frac{\mathfrak{N}\{X(i,\epsilon)\}}{N^{(\alpha+\eta)i}} \to 0 \quad (i \to \infty). \tag{3}$$

Let
$$K = 4 \max_{i,j=0,1,\ldots,N-1} \frac{p_i}{p_j}, \tag{4}$$

and ϵ satisfy
$$0 < \epsilon < \min_{i=0,1,\ldots,N-1} \tfrac{1}{2}p_i, \tag{5}$$

and
$$K^{2N\epsilon} < N^{\eta}. \tag{5'}$$

Choose an integer i_0 so that

$$\epsilon i_0 > \max\{1, \tfrac{1}{2}(N-1)\}. \tag{6}$$

Now
$$\begin{aligned}\mathfrak{N}\{X(i,\epsilon)\} &= \sum {}^{i}C_{t_1}\, {}^{i-t_1}C_{t_2} \ldots {}^{i-t_1-t_2-\ldots-t_{N-2}}C_{t_{N-1}} \\ &= \sum \frac{i!}{t_0!\, t_1! \ldots t_{N-1}!},\end{aligned} \tag{7}$$

where the summation is over sets of integers $t_0, t_1, \ldots, t_{N-1}$ such that

$$\sum_{j=0}^{N-1} t_j = i \quad \text{and} \quad i(p_j-\epsilon) \leqslant t_j \leqslant i(p_j+\epsilon) \quad (j = 0, 1, \ldots, N-1).$$

Let $\quad Q_1 = \dfrac{i!}{t_0!\, t_1! \ldots t_{N-1}!}, \qquad Q_2 = \dfrac{i!}{(t_0+1)!\, (t_1-1)! \ldots t_{N-1}!}$

be two terms of the sum (7) for $i \geqslant i_0$.

Then
$$\begin{aligned}\frac{Q_1}{Q_2} = \frac{t_0+1}{t_1} &\leqslant \frac{(p_0+\epsilon)i+1}{(p_1-\epsilon)i} \\ &< \frac{2p_0}{\frac{1}{2}p_1}, \quad \text{by (5) and (6)}, \\ &< K.\end{aligned} \tag{8}$$

Let Q_0 be the term of (7) for which t_j is $p'_j i$, the integral part of $p_j i$, for $j = 1, 2, \ldots, N-1$, and $t_0 = i - t_1 - t_2 - \ldots - t_{N-1}$.

Then, for any term Q of (7),

$$\frac{Q}{Q_0} < (K)^{2N\epsilon i}. \tag{9}$$

By (9), $\quad \mathfrak{N}\{X(i,\epsilon)\} < Q_0 K^{2N\epsilon i}(2\epsilon i)^{N-1}.$

But

$$Q_0 \sim \frac{K_1 i^{i+\frac{1}{2}}}{i^{i+\frac{1}{2}N} \prod_{j=0}^{N-1} (p'_j)^{p'_j i+\frac{1}{2}}} \tag{10}$$

$$\leqslant K_2 i^{\frac{1}{2}(1-N)} N^{\alpha i},$$

where K_1, K_2 are appropriate positive constants.

Thus

$$\frac{\mathfrak{N}\{X(i,\epsilon)\}}{N^{(\alpha+\eta)i}} \leqslant \frac{K_2 i^{\frac{1}{2}(1-N)} N^{\alpha i} K^{2N\epsilon i}(2\epsilon i)^{N-1}}{N^{(\alpha+\eta)i}} \to 0 \quad \text{as} \quad i \to \infty, \text{ by } (5').$$

Thus (3) is established, and (*a*) follows.

(*b*) *Dimension of $S \geqslant \alpha$.* The method is to show that, given $\eta > 0$, there is a closed set S_1 with the properties:

(i) $S_1 \subset S$;

(ii) any set of intervals U for which $\sum_U d^{\alpha-\eta} < 1$ and of which the maximum diameter of all members is sufficiently small does not cover the whole of S_1.

These imply that the $(\alpha-\eta)$ measure of S_1 and *a fortiori* that of S is not less than 1. Since this is true of all $\eta > 0$, it follows that the dimension of S is not less than α. I shall suppose η such that $\alpha > \eta$.

The same notation is used as in (*a*); in addition $Y(i)$ is used to denote those i-figured decimals for which the digit j occurs $p'_j i$ times for $j = 1, 2, \ldots, N-1$. (The number of such decimals is denoted above by Q_0.) Let $\{\epsilon_i\}$ and $\{\eta_i\}$ denote two sequences of positive decreasing numbers tending to zero and such that $\{\eta_i\}$ satisfies the additional condition $\sum_{j=1}^{\infty} \eta_j < \frac{1}{2}$.

I need the following lemma which is a generalization of one due to A. S. Besicovitch [(1) Lemma 2].

To each pair (ϵ_i, η_i) there corresponds a positive integer l_i such that, of all the decimals $Y(k)$ with $k \geqslant l_i$, more than $(1-\eta_i)\mathfrak{N}\{Y(k)\}$ satisfy

$$(p_r - \epsilon_i)j \leqslant P(x,j,r) \leqslant (p_r + \epsilon_i)j \tag{11}$$

for each integer j satisfying $l_i \leqslant j \leqslant k$, and $r = 0, 1, 2, \ldots, N-1$, i.e. $\prod_{j=l_i}^{k} X(j, \epsilon_i)$ contains more than $(1-\eta_i)\mathfrak{N}\{Y(k)\}$ of the decimals forming $Y(k)$.

By (10), there is a positive integer M such that, for $k \geqslant M$,

$$\mathfrak{N}\{Y(k)\} \geqslant \frac{K_3 N^{\alpha k}}{k^{\frac{1}{2}(N-1)}}, \tag{12}$$

where K_3 is some positive constant.

Let m be a given integer $\geqslant \max\{l_1, M\} = M_0$; then there is a largest integer m' such that

$$m \geqslant l_{m'} \quad \text{and} \quad m\epsilon_{m'} > 1.$$

Moreover, as $m \to \infty$, so does m'.

Write $$S_1 = \prod_{m=M_0}^{\infty} X(m, \epsilon_{m'}),$$

and denote $X(m, \epsilon_{m'})$ by $X(m)$.

For $k \geqslant M_0$, $\prod_{m=M_0}^{k} X(m)$ contains more than $\frac{1}{2}\mathfrak{N}\{Y(k)\}$ of the decimals $Y(k)$. Clearly S_1 is closed and has property (i). In order to complete the proof of the theorem, it is sufficient to show that it has property (ii). Because S_1 is closed, it is sufficient to establish property (ii) for finite coverings U only.

Let U be any finite set of intervals for which

$$\sum_U d^{\alpha-\eta} < 1. \tag{13}$$

For any interval L of U, let $l = l(L)$ be the positive integer such that

$$\frac{1}{N^l} > d(L) \geqslant \frac{1}{N^{l+1}}.$$

Associate with L the least closed interval L' for which

$$L \supset L' \supset LS_1.$$

Both end-points of L' belong to S_1 and thus L' is contained in either one interval, or two abutting intervals of $X(l)$. In the first case, let the interval be I, and, in the second, let the two intervals be I' and I''. Then either,

$$(dL)^{\alpha-\eta} \geqslant \left(\frac{1}{N}\right)^{\alpha-\eta} (dI)^{\alpha-\eta},$$

or $$(dL)^{\alpha-\eta} \geqslant \frac{1}{2N^{\alpha-\eta}}\{(dI')^{\alpha-\eta} + (dI'')^{\alpha-\eta}\}.$$

The totality of all such intervals I or I' and I'' derived from all the L of U is called W. W has the following properties:

(i) it consists of a finite set of intervals;

(ii) it contains all that part of S_1 contained in U;

(iii) $\sum_W d^{\alpha-\eta} \leqslant 2N^{\alpha-\eta} \sum_U d^{\alpha-\eta} < 2N^{\alpha-\eta}$;

(iv) every interval of W belongs to an interval of $X(l)$ for some integer l;

(v) as the diameter of the maximum interval of U decreases to zero, the least of the integers l defined in (iv) above increases to infinity.

Now suppose that I_1 and I_2 are any two intervals of $X(l)$ with left-hand end-points x_1, x_2, and that $N_1^{(i)}$, $N_2^{(i)}$ are the number of $Y(i)$ which belong to them respectively. Let x_1, x_2, when expressed as decimals in the scale of N, contain the digit j, r_j' and r_j'' times respectively ($j = 0, 1, \ldots, N-1$).

For $i > l$

$$\frac{N_1^{(i)}}{N_2^{(i)}} = \prod_{j=0}^{N-1} \frac{(p_j' i - r_j'')!}{(p_j' i - r_j')!}$$

$$\sim \prod_{j=0}^{N-1} \left(\frac{p_j' i - r_j''}{p_j' i - r_j'}\right)^{p_j' i - r_j'} \left(\frac{p_j' i - r_j''}{p_j' i - r_j'}\right)^{\frac{1}{2}} (p_j' i - r_j'')^{r_j' - r_j''}$$

$$\sim \prod_{j=0}^{N-1} (p_j')^{r_j' - r_j''}.$$

Thus, if i is large compared with l, there is a constant A such that

$$A^{-\epsilon(l)l} < \frac{N_1^{(i)}}{N_2^{(i)}} < A^{\epsilon(l)l}, \quad \text{where} \quad \epsilon(l) = \epsilon_{l'}.$$

If $\overline{N}$ is the maximum number of $Y(i)$ inside any one interval of $X(l)$, then

$$\overline{N} < \frac{A^{l\epsilon(l)} \mathfrak{N}\{Y(i)\}}{\mathfrak{N}\{X(l)\}}.$$

Hence the total number of $Y(i)$ covered by U

$$\leqslant \text{total number of } Y(i) \text{ covered by } W$$

$$\leqslant \sum_{L \in W} \frac{A^{\epsilon(l)l} \mathfrak{N}\{Y(i)\}}{\mathfrak{N}\{X(l)\}}$$

$$\leqslant \mathfrak{N}\{Y(i)\} \sum_W \frac{A^{\epsilon(l)l} l^{\frac{1}{2}(N-1)}}{K_3 N^{\alpha l}}, \quad \text{by (12).}$$

There is an integer l_0 such that, for $l \geqslant l_0$,

$$\frac{A^{\epsilon(l)l\frac{1}{2}(N-1)}}{K_3 N^{\eta l}} < \frac{1}{3.2.N^{\alpha-\eta}}.$$

Then the total number of $Y(i)$ covered by U is

$$\leqslant \frac{1}{3}\frac{\mathfrak{N}\{Y(i)\}}{2N^{\alpha-\eta}}\sum_{W}\frac{1}{N^{l(\alpha-\eta)}} = \frac{1}{3}\frac{\mathfrak{N}\{Y(i)\}}{2N^{\alpha-\eta}}\sum_{W} d^{\alpha-\eta} < \tfrac{1}{3}\mathfrak{N}\{Y(i)\},$$

provided that the maximum diameter of the members of U is so small that only intervals of $X(l)$ with $l \geqslant l_0$ appear in W.

Thus there are intervals of $X(i)$ contained in $\prod_{m=M_0}^{i} X(m)$ and not covered by U for $i \geqslant i_2$ (say). Let these be $P(i)$. Then

$$\prod_{i=i_2}^{\infty} P(i)$$

is a closed non-void subset of S_1 not covered by U. This establishes property (ii) and part (*b*) of the theorem follows immediately.

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A DIVISOR PROBLEM

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1. LET $\sigma(n)$ denote the sum of the divisors of n. The error terms $R(x)$, $R_1(x)$ in the approximate formulae

$$\sum_{n\leqslant x}\frac{1}{n}\sigma(n) = \tfrac{1}{6}\pi^2 x-\tfrac{1}{2}\log x+R(x),$$

$$\sum_{n\leqslant x}\sigma(n) = \tfrac{1}{12}\pi^2 x^2+xR_1(x)$$

were investigated by Walfisz.† If

$$\rho(x) = \sum_{m\leqslant x}\frac{1}{m}\psi\left(\frac{x}{m}\right), \tag{1}$$

where $\psi(u)$ is defined as $u-[u]-\frac{1}{2}$ for all real u, it is easily seen that‡

$$R(x) = -\rho(x)+O(1), \qquad R_1(x) = -\rho(x)+O(1).$$

It is obvious that $\rho(x) = O(\log x)$, and Walfisz improved this, by the use of Weyl's inequality for exponential sums, to

$$\rho(x) = O\left(\frac{\log x}{\log\log x}\right).$$

Further progress is now possible because of Vinogradov's remarkable improvements on Weyl's inequality. In this note, I apply one of Vinogradov's inequalities in the form given by Titchmarsh,§ and prove that

$$\rho(x) = O\{(\log x)^{\frac{4}{5}+\epsilon}\} \tag{2}$$

for any $\epsilon > 0$.

Except in (2) and in the final stage of the proof, the constants implied by the symbol O will always be *absolute* constants.

2. LEMMA 1. *Let* $F(x) = \alpha x^n+\alpha_1 x^{n-1}+...+\alpha_n$, *where* $n \geqslant 3$ *and* $\alpha, \alpha_1,..., \alpha_n$ *are real*, $\alpha \neq 0$. *Let P be a positive integer and suppose that*

$$2nP|\alpha| \leqslant 1. \tag{3}$$

Then

$$\sum_{m=1}^{P} e^{2\pi i F(m)} = O(nP^{1-\rho}+|\alpha|^{-1/(n-1)}), \tag{4}$$

† A. Walfisz, 'Teilerprobleme', *Math. Zeits.* 26 (1927), 66–88.

‡ A proof is given in Lemma 1 of Walfisz, loc. cit.

§ E. C. Titchmarsh, 'On $\zeta(s)$ and $\pi(x)$', *Quart. J. of Math.* (Oxford), 9 (1938), 97–108. This is based on the method used in Vinogradov, *Recueil Math.* (Moscow), N.S. 1 (43) (1936), 9–19.

where
$$\rho = \rho_n = \frac{A}{n^4 \log^2 n}, \tag{5}$$
A being a certain positive absolute constant.

Proof. This is the inequality of Vinogradov already referred to and is equation (8.1) of Titchmarsh, loc. cit. We have to replace n there by $n-1$, but there is obviously no need to do this explicitly in the formula for ρ.

LEMMA 2. *Let $n \geqslant 3$ be an integer; let P, Q be positive integers and let z be a positive real number. Suppose that*
$$P \leqslant \tfrac{1}{2}Q, \tag{6}$$
$$2nzP \leqslant Q^{n+1}, \tag{7}$$
$$zP^{n+1} \leqslant Q^{n+2}. \tag{8}$$
Then
$$\sum_{m=Q+1}^{Q+P} e^{2\pi i z/m} = O\left\{nP^{1-\rho} + \left(\frac{Q^{n+1}}{z}\right)^{1/(n-1)}\right\}. \tag{9}$$

Proof. If $1 \leqslant m \leqslant P$, we have, by (6),
$$\frac{z}{Q+m} = \sum_{\nu=0}^{n} \frac{(-1)^\nu z m^\nu}{Q^{\nu+1}} + \sum_{\nu=n+1}^{\infty} \frac{(-1)^\nu z m^\nu}{Q^{\nu+1}} = F(m) + G(m),$$
say. Here $F(m)$ is a polynomial in m of degree n whose highest coefficient is
$$\alpha = \frac{(-1)^n z}{Q^{n+1}}. \tag{10}$$

Let
$$e^{2\pi i G(m)} = \sum_{k=0}^{\infty} c_k m^k.$$
Then
$$\sum_{k=0}^{\infty} |c_k| P^k \leqslant \exp\left(2\pi \sum_{\nu=n+1}^{\infty} \frac{zP^\nu}{Q^{\nu+1}}\right) \leqslant \exp\left(\frac{4\pi z P^{n+1}}{Q^{n+2}}\right),$$
by (6). It follows from (8) that
$$\sum_{k=0}^{\infty} |c_k| P^k = O(1). \tag{11}$$

The sum on the left of (9) is
$$\sum_{m=1}^{P} e^{2\pi i z/(Q+m)} = \sum_{m=1}^{P} e^{2\pi i F(m)} \sum_{k=0}^{\infty} c_k m^k. \tag{12}$$
The condition (3) of Lemma 1 is satisfied, by (7) and (10), and moreover is satisfied *a fortiori* if P is replaced by any smaller positive integer. Hence
$$\sum_{m=1}^{P'} e^{2\pi i F(m)} = O\left\{nP^{1-\rho} + \left(\frac{Q^{n+1}}{z}\right)^{1/(n-1)}\right\}$$

for $1 \leqslant P' \leqslant P$. It follows by partial summation that

$$\sum_{m=1}^{P} m^k e^{2\pi i F(m)} = O\left[P^k\left\{nP^{1-\rho}+\left(\frac{Q^{n+1}}{z}\right)^{1/(n-1)}\right\}\right]$$

for $k \geqslant 0$. Substituting in (12) and using (11), we obtain (9).

LEMMA 3. *Let n be an integer greater than n_0, a suitable absolute constant. Suppose that $z > 2^{2n}$, and that Q, Q' are integers satisfying*

$$(4n^2z)^{1/(n+1)} < Q < z^{2/(n+4)}, \qquad Q < Q' \leqslant 2Q. \tag{13}$$

Then
$$\sum_{m=Q+1}^{Q'} e^{2\pi i z/m} = O\left\{nQ\left(\frac{Q^{n+1}}{z}\right)^{-\eta}\right\}, \tag{14}$$

where
$$\eta = \eta_n = \frac{A}{n^5 \log^2 n}. \tag{15}$$

Proof. We choose $n_0 > 8$ so that

$$(n-1)(1-\rho) > n-2, \tag{16}$$

$$\frac{2+(n-1)\rho}{n+3+(n+2)(n-1)\rho} > \frac{2}{n+4} \tag{17}$$

for $n > n_0$. We define P to be the largest integer for which

$$P^{(n-1)(1-\rho)} \leqslant \frac{Q^{n+1}}{z}, \tag{18}$$

and assert that z, P, Q satisfy the hypotheses of Lemma 2. In the first place, $P \geqslant 1$ since $Q^{n+1} > 4n^2z$. Next, (6) is satisfied since

$$(\tfrac{1}{2}Q)^{(n-1)(1-\rho)} > (\tfrac{1}{2}Q)^{n-2} > \frac{Q^{n+1}}{z}$$

by (16) and by
$$Q^3 < z^{6/(n+4)} < \frac{z}{2^n}.$$

Also (7) is satisfied; for this it suffices to prove that

$$\left(\frac{Q^{n+1}}{2nz}\right)^{(n-1)(1-\rho)} > \frac{Q^{n+1}}{z},$$

and, since $(n-1)(1-\rho) > 2$, this follows from $Q^{n+1} > 4n^2z$. Finally, we prove that (8) is satisfied. This will be so if

$$\left(\frac{Q^{n+2}}{z}\right)^{(n-1)(1-\rho)/(n+1)} > \frac{Q^{n+1}}{z},$$

i.e.
$$Q^{n+3+(n+2)(n-1)\rho} < z^{2+(n-1)\rho},$$

which follows from (17) and (13).

Since the conditions of Lemma 2 are satisfied by P and Q, they will continue to be satisfied if Q is replaced by a larger number and P by a smaller number. Now the sum in (14) can be dissected into sums of the form

$$\sum_{m=Q_r+1}^{Q_r+P} e^{2\pi iz/m},$$

where $Q_r = Q+rP$ and $r = 0, 1, 2,\ldots$, together with a similar sum of less than P terms. The number of sums is $O(Q/P)$, and, by Lemma 2, each of them is

$$O\left\{nP^{1-\rho}+\left(\frac{(2Q)^{n+1}}{z}\right)^{1/(n-1)}\right\},$$

since $Q_r < 2Q$. The sum in (14) is therefore

$$O\left\{nQP^{-\rho}+QP^{-1}\left(\frac{Q^{n+1}}{z}\right)^{1/(n-1)}\right\} = O\left\{nQ\left(\frac{Q^{n+1}}{z}\right)^{-\eta}\right\},$$

by the definition of P in (18), where

$$\eta = \frac{\rho}{(n-1)(1-\rho)} > \frac{A}{n^5\log^2 n},$$

and so can be replaced by the value stated in (15).

LEMMA 4. *Let x be a positive real number, and let Q, Q' be positive integers satisfying $Q < Q' \leqslant 2Q$. Then, for any positive integer N,*

$$\left|\sum_{m=Q+1}^{Q'} \psi\left(\frac{x}{m}\right)\right| < \sum_{r=1}^{N} \frac{1}{r}\left|\sum_{m=Q+1}^{Q'} e^{2\pi irx/m}\right| + 2QN^{-\frac{1}{2}}. \qquad (19)$$

Proof. I follow Walfisz (loc. cit., Lemma 4) and use one of the older devices of analytic number-theory, first presented in this form by van der Corput. For $h > 0$ we have

$$\psi(t) \geqslant \psi(t+h)-h,$$

whence, by integration,

$$\psi(t) \geqslant \frac{1}{\delta}\int_0^\delta \psi(t+h)\,dh-\tfrac{1}{2}\delta,$$

for any $\delta > 0$. Similarly

$$\psi(t) \leqslant \frac{1}{\delta}\int_0^\delta \psi(t-h)\,dh+\tfrac{1}{2}\delta.$$

Hence, if we obtain any estimate which is valid for both the expressions

$$\frac{1}{\delta}\int_0^\delta \sum_{m=Q+1}^{Q'} \psi\left(\frac{x}{m}\pm h\right) dh,$$

and add to it $\frac{1}{2}\delta Q$, we obtain a valid estimate for the sum on the left of (19).

It is well known that

$$\psi(t) = -\sum_{r=-\infty}^{\infty}{}' \frac{e^{2\pi irt}}{2\pi ir},$$

where the term $r = 0$ is to be omitted and the terms r, $-r$ are always to be taken together. Hence

$$\frac{1}{\delta}\int_0^\delta \psi(t+h)\, dh = \sum_{r=-\infty}^{\infty}{}' a_r e^{2\pi irt},$$

where $a_0 = 0$ and

$$|a_r| \leqslant \min\left(\frac{1}{2\pi|r|}, \frac{2}{(2\pi r)^2\delta}\right)$$

for $r \neq 0$. It follows that

$$\left|\frac{1}{\delta}\int_0^\delta \sum_{m=Q+1}^{Q'} \psi\left(\frac{x}{m}+h\right) dh\right| \leqslant \sum_{r=1}^{N} \frac{1}{r}\left|\sum_{m=Q+1}^{Q'} e^{2\pi irx/m}\right| + \sum_{r=N+1}^{\infty} \frac{Q}{\delta r^2}$$

for any positive integer N.

We choose $\delta = N^{-\frac{1}{2}}$; then the second expression on the right is less than $QN^{-\frac{1}{2}}$. A similar treatment applies to the integral with $-h$ in place of h, and, on adding the earlier term $\frac{1}{2}\delta Q = \frac{1}{2}QN^{-\frac{1}{2}}$, we obtain the result stated.

Lemma 5. *Let n be a positive integer greater than n_1, a suitable absolute constant. Suppose that $x > 2^{2n}$ and that Q, Q' are integers satisfying*

$$x^{2/(n+5)} < Q < x^{2/(n+4)}, \qquad Q < Q' \leqslant 2Q. \tag{20}$$

Then

$$\sum_{m=Q+1}^{Q'} \psi\left(\frac{x}{m}\right) = O(n^6 \log^2 n\, Qx^{-\zeta}), \tag{21}$$

where

$$\zeta > \frac{A}{2n^5 \log^2 n}. \tag{22}$$

Proof. By Lemma 4, the absolute value of the sum in (21) is less than

$$\sum_{r=1}^{N} \frac{1}{r}\left|\sum_{m=Q+1}^{Q'} e^{2\pi irx/m}\right| + 2QN^{-\frac{1}{2}},$$

where N is any positive integer. Each of the inner sums here will satisfy the conditions of Lemma 3, with $z = rx$, provided

$$4n^2Nx < Q^{n+1}. \tag{23}$$

Subject to this condition, we obtain the estimate

$$O\left\{\sum_{r=1}^{N} \frac{1}{r} nQ\left(\frac{Q^{n+1}}{rx}\right)^{-\eta} + QN^{-\frac{1}{2}}\right\}.$$

Now
$$\sum_{r=1}^{N} r^{-1+\eta} = O(\eta^{-1}N^{\eta});$$

hence the above estimate is

$$O\{n^6 \log^2 n\, Q^{1-(n+1)\eta} x^{\eta} N^{\eta} + QN^{-\frac{1}{2}}\}. \tag{24}$$

We choose N by
$$N = \left[\left(\frac{Q^{n+1}}{x}\right)^{2\eta/(2\eta+1)}\right], \tag{25}$$

and note that
$$\frac{Q^{n+1}}{x} > x^{(n-3)/(n+5)} \tag{26}$$

by (20). The condition (23) is satisfied if

$$4n^2\left(\frac{Q^{n+1}}{x}\right)^{2\eta/(2\eta+1)} < \frac{Q^{n+1}}{x},$$

i.e.
$$\frac{Q^{n+1}}{x} > (4n^2)^{2\eta+1}.$$

This follows from (26) with $x > 2^{2n}$, provided that the absolute constant n_1 is suitably chosen.

Substituting for N from (25) in (24), we obtain

$$O\left\{n^6 \log^2 n\, Q\left(\frac{Q^{n+1}}{x}\right)^{-\eta/(2\eta+1)}\right\} = O(n^6 \log^2 n\, Qx^{-\zeta}),$$

by (26), where
$$\zeta = \frac{(n-3)\eta}{(n+5)(2\eta+1)} > \frac{A}{2n^5 \log^2 n},$$

if n_1 is suitably chosen.

LEMMA 6. *Suppose that $x > 2^{2n}$ and $n > n_1$. Then*

$$\sum \frac{1}{m} \psi\left(\frac{x}{m}\right) = O(n^6 \log^2 n\, x^{-\zeta} \log x), \tag{27}$$

where the summation is taken over

$$x^{2/(n+5)} < m \leqslant x^{2/(n+4)}.$$

Proof. If Q, Q' satisfy the conditions of Lemma 5, partial summation from (21) gives

$$\sum_{m=Q+1}^{Q'} \frac{1}{m}\psi\left(\frac{x}{m}\right) = O(n^6 \log^2 n\, x^{-\zeta}).$$

The sum in (27) can be dissected into $O(s)$ sums of this kind, where s is the largest integer for which

$$2^s < x^{\{2/(n+4)\}-\{2/(n+5)\}}.$$

Since $s = O(\log x)$, the result follows.

3. We require also two older lemmas. The first of them is Walfisz's principal lemma (loc. cit., Lemma 4). I have altered his notation slightly.

LEMMA 7. *Suppose that $x \geqslant 3$, and $n \geqslant 0$ is an integer. Then*

$$\sum \frac{1}{m}\psi\left(\frac{x}{m}\right) = O(x^{-\kappa} \log^2 x),$$

where the summation is taken over

$$x^{2/(n+5)} < m \leqslant x^{2/(n+4)}$$

and
$$\kappa = \frac{1}{20(n+1)2^n}.$$

The second lemma is due to Wigert,† and is of a more elementary character.

LEMMA 8.
$$\sum_{\sqrt{x}<m\leqslant x} \frac{1}{m}\psi\left(\frac{x}{m}\right) = O(1).$$

4. The proof of (2) now follows immediately. By Lemmas 7 and 8, we have

$$\sum \frac{1}{m}\psi\left(\frac{x}{m}\right) = O(1),$$

the summation being taken over

$$x^{2/(n_1+5)} < m \leqslant x,$$

where n_1 is any absolute constant. If $n > n_1$ and $x > 2^{2n}$, addition of $n-n_1$ sums of the type considered in Lemma 6 gives us

$$\sum \frac{1}{m}\psi\left(\frac{x}{m}\right) = O(n^7 \log^2 n\, x^{-\zeta} \log x), \tag{28}$$

† S. Wigert, *Acta Math.* 37 (1913), 113–40. The result follows from the argument on p. 118.

the summation being taken over

$$x^{2/(n+5)} < m \leqslant x^{2/(n_1+5)},$$

where
$$\zeta > \frac{A}{2n^5 \log^2 n}.$$

We choose n as a function of x so that

$$n \sim (\log x)^{1/k}$$

as $x \to \infty$, where $k > 5$, and then the right-hand side of (28) is $O(1)$, where the constant implied by the symbol O now depends on k. Finally,

$$\sum \frac{1}{m} \psi\left(\frac{x}{m}\right) = O\left(\frac{1}{n} \log x\right) = O\{(\log x)^{1-1/k}\},$$

the summation being taken over

$$1 \leqslant m < x^{2/(n+5)}.$$

This completes the proof of (2).

[*Added* 4 *March* 1949.] Professor A. Walfisz has kindly drawn my attention to the fact that he has already proved this result, in a slightly more precise form, in his paper 'Über Gitterpunkte in mehrdimensionalen Ellipsoiden, achte Abhandlung', *Travaux de l'Inst. Math. de Tbilissi*, 5 (1938), 181–96. I regret having overlooked this paper. He also tells me that the estimate for $\rho(x)$ can probably be improved to $O\{(\log x)^{\frac{2}{3}+\epsilon}\}$ by making use of later inequalities of Vinogradov.

ON THE CRITICAL DETERMINANT OF A CERTAIN NON-CONVEX CYLINDER

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1. We use K to denote any two-dimensional star body in the xy-plane, and C to denote the corresponding three-dimensional cylindrical star body consisting of those points P with coordinates (x, y, z) such that (x, y) lies in K and $|z| \leqslant 1$. Following Mahler,* the critical determinant $\Delta(S)$ of a star body S is the lower bound of the determinants of the lattices Λ admissible for S, i.e. the lattices Λ with no point other than the origin O in the interior of S.

Yeh† and Chalk and I‡ have proved that, if K is convex, then

$$\Delta(C) = \Delta(K). \tag{1}$$

Varnavides§ has shown that (1) also holds in the particular case when K is the non-convex star body given by

$$|xy| \leqslant 1.$$

It is easy to see that in general

$$\Delta(C) \leqslant \Delta(K).$$

Dr. Mahler has suggested that it would be of interest to give an example of a star body K such that

$$\Delta(C) < \Delta(K).$$

The object of this note is to give such an example.

2. Let A, B, C, D, E be the points with coordinates

$$(1,0), \quad (1,\epsilon), \quad (\tfrac{3}{2}, \tfrac{1}{2}-\tfrac{1}{2}\epsilon), \quad \left(\frac{2-2\epsilon}{1-2\epsilon}, \frac{1-\epsilon}{1-2\epsilon}\right), \quad (0, 1-\epsilon),$$

where $0 < \epsilon < \frac{1}{4}$. Let A', B', C', D', E' be the images of these points in the y-axis. We take K to be the star body bounded by the line

* K. Mahler, *Proc. Royal Soc.* A, 187 (1946), 151–87.
† Y. Yeh, *J. of London Math. Soc.* 23 (1948), 188–95.
‡ J. H. H. Chalk and C. A. Rogers, ibid. 178–87.
§ P. Varnavides, ibid. 195–9.

segments joining the points A, B, C, D, E, D', C', B', $-A$, $-B$, $-C$, $-D$, $-E$, $-D'$, $-C'$, $-B'$, and A. (See Fig. 1.) I prove that

$$\Delta(C) \leqslant 1 < \frac{(1-\epsilon)^2}{1-2\epsilon} \leqslant \Delta(K). \tag{2}$$

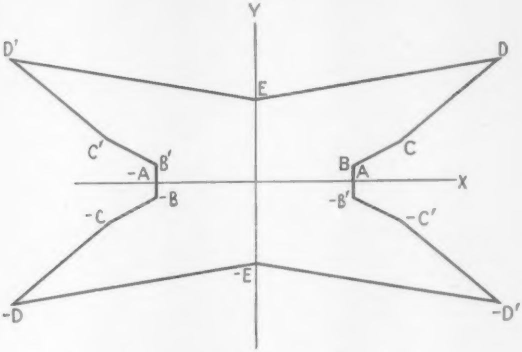

FIG. 1. The star body K, with $\epsilon = \frac{1}{8}$.

It is easy to verify that the lattice generated by the points

$$(2, 0, 0), \quad (1, 1, 0), \quad (1, \epsilon, \tfrac{1}{2})$$

is admissible for C. As the determinant of this lattice is 1, it follows that

$$\Delta(C) \leqslant 1.$$

To complete the proof of (2) I apply Mordell's method* to the body K. We suppose that Λ is a lattice with determinant

$$\Delta < \frac{(1-\epsilon)^2}{1-2\epsilon} \tag{3}$$

and prove that there is a lattice point of Λ other than O inside K.

The parallelogram $-E$, D, E, $-D$ has area

$$4\frac{(1-\epsilon)^2}{1-2\epsilon} > 4\Delta.$$

So by Minkowski's theorem there is a point P other than O of Λ inside this parallelogram. If Λ is admissible for K, the point P must lie in one of the triangles A, B, C or $-A$, $-B$, $-C$, not on the side A, C or the side $-A$, $-C$. Then there is a point P_1 of Λ in the triangle

* See, for example, L. J. Mordell, *Proc. London Math. Soc.* (2), 48 (1943), 198–228.

A, B, C not on the side A, C. Similarly there is a point P_2 of Λ in the triangle A, $-B'$, $-C'$, not on the side A, $-C'$. It is easy to verify that the point $P_2 - P_1$ is a point of Λ other than O in the interior of K. Thus every lattice Λ which is admissible for K has a determinant not less than

$$\frac{(1-\epsilon)^2}{1-2\epsilon}.$$

Hence
$$\Delta(K) \geqslant \frac{(1-\epsilon)^2}{1-2\epsilon}.$$

This completes the proof of (2) and shows that K has the required property.

I remark that as $\epsilon \to \frac{1}{4}$

$$\frac{(1-\epsilon)^2}{1-2\epsilon} \to \frac{9}{8}.$$

The above argument is still valid when $\epsilon = \frac{1}{4}$, but then K is not a star body in Mahler's sense.

AN IMPROVEMENT OF VINOGRADOV'S MEAN-VALUE THEOREM AND SEVERAL APPLICATIONS

By LOO-KENG HUA

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1. Introduction. IN 1940 I showed† that Vinogradov's estimation (1) of Weyl's sum depends essentially on a result which I called 'Vinogradov's mean-value theorem'. The purpose of the present paper is to improve this mean-value theorem. Since Vinogradov's method seems to have reached a final stage, any improvement of the constant in the exponent is worthy of consideration. More definitely, in this paper I shall establish a mean-value theorem by means of which we can establish sharper results about Waring's problem, distribution of primes, etc. The method used here seems to be much simpler than that originally used by Vinogradov.

The form of Vinogradov's mean-value theorem which will be proved here is the following.

THEOREM 1. *Let P and T be integers and $P \geqslant 2$,*

$$f(x) = \alpha_k x^k + \ldots + \alpha_1 x, \tag{1}$$

and let

$$C_k = C_k(P) = \sum_{T<x\leqslant T+P} e^{2\pi i f(x)}. \tag{2}$$

Then, when

$$s \geqslant \tfrac{1}{4}k(k+1)+lk, \tag{3}$$

we have

$$\int_0^1 \ldots \int_0^1 |C_k(P)|^{2s}\, d\alpha_1 \ldots d\alpha_2 \leqslant (7s)^{4sl}(\log P)^{2l} P^{2s-\frac{1}{2}k(k+1)+\delta}, \tag{4}$$

where

$$\delta = \tfrac{1}{2}k(k+1)(1-1/k)^l. \tag{5}$$

Arithmetically, the value of the integral in (4) is equal to the number of solutions of the system of equations

$$x_1^h + \ldots + x_s^h = y_1^h + \ldots + y_s^h \quad (1 \leqslant h \leqslant k), \tag{6}$$

where

$$T < x_i, \quad y_i \leqslant T+P.$$

† Hua, *Additive Prime Number Theory*. This booklet was accepted for publication by the Academy of U.S.S.R. in 1940, but its appearance was delayed by the war.

Setting $X_i = x_i - T$, $Y_i = y_i - T$, we obtain from (6)

$$\sum_{i=1}^{s} (X_i + T)^h = \sum_{i=1}^{s} (Y_i + T)^h \quad (1 \leqslant h \leqslant k). \tag{7}$$

Expanding the hth powers, we see that the system of equations (7) is equivalent to

$$\sum_{i=1}^{s} X_i^h = \sum_{i=1}^{s} Y_i^h \quad (1 \leqslant h \leqslant k), \tag{8}$$

where $0 < X_i, Y_i \leqslant P$. This establishes that the left-hand side of (4) is really independent of T.

Since Vinogradov's paper and my booklet are both published in Russian, the paper is set forth *ab initio*.

2. Lemmas

LEMMA 1. *Let $Q = RH$, $R > 1$, $H > 1$ and let $g_1,\ldots, g_k$ be integers satisfying*

$$1 < g_1 < g_2 < \ldots < g_k \leqslant H, \qquad g_\nu - g_{\nu-1} > 1. \tag{9}$$

For each value of ν $(1 \leqslant \nu \leqslant k)$ let x_ν be a variable lying in the interval

$$-\omega + (g_\nu - 1)R < x_\nu \leqslant -\omega + g_\nu R \quad (0 \leqslant \omega \leqslant Q). \tag{10}$$

The number of sets of such integers $x_1,\ldots, x_k$ for which the values of

$$x_1^h + \ldots + x_k^h \tag{11}$$

lie in intervals of lengths not exceeding Q^{h-1} $(1 \leqslant h \leqslant k)$ respectively, is less than or equal to

$$(2kH)^{\frac{1}{2}k(k-1)}. \tag{12}$$

Proof. Let $x_1,\ldots, x_k$ and $y_1,\ldots, y_k$ be two sets of integers satisfying the requirements of the lemma; let

$$s_h = \sum_{\nu=1}^{k} x_\nu^h, \qquad s_h' = \sum_{\nu=1}^{k} y_\nu^h,$$

and let σ_h be the hth elementary symmetric function of $x_1,\ldots, x_k$ and σ_h' that of $y_1,\ldots, y_k$. Then, by (10), we have

$$|s_h| \leqslant \sum_{\nu=1}^{k} |x_\nu|^h \leqslant kQ^h, \qquad |s_h'| \leqslant kQ^h \tag{13}$$

and

$$|\sigma_h| \leqslant \binom{k}{h} Q^h, \qquad |\sigma_h'| \leqslant \binom{k}{h} Q^h. \tag{14}$$

By the hypotheses, we have

$$|s_h - s_h'| \leqslant Q^{h-1} \quad (1 \leqslant h \leqslant k). \tag{15}$$

From (15), we shall deduce that

$$|\sigma_h - \sigma_h'| \leqslant \tfrac{3}{4}(2kQ)^{h-1} \quad \text{for} \quad 2 \leqslant h \leqslant k, \tag{16}$$

and therefore

$$|\sigma_h-\sigma_h'| \leqslant (2kQ)^{h-1} \quad \text{for} \quad 1 \leqslant h \leqslant k. \tag{17}$$

Since $\sigma_2 = \frac{1}{2}(s_1^2-s_2)$, we have by (13)

$$|\sigma_2-\sigma_2'| \leqslant \tfrac{1}{2}\{|s_1-s_1'|(|s_1|+|s_1'|)+2|s_2-s_2'|\} \leqslant \tfrac{1}{2}(2k+1)Q \leqslant \tfrac{3}{4}(2kQ), \tag{18}$$

so that (16) holds for $h = 2$. We use induction and suppose that (16) is true for $2 \leqslant h \leqslant t-1$. We then deduce from (13), (14), (15), and (16) that, for $1 \leqslant \nu \leqslant t-1$,

$$\begin{aligned}|\sigma_\nu s_{t-\nu}-\sigma_\nu' s_{t-\nu}'| &\leqslant |\sigma_\nu-\sigma_\nu'||s_{t-\nu}|+|\sigma_\nu'||s_{t-\nu}-s_{t-\nu}'| \\ &\leqslant \left\{(2k)^\nu k+\binom{k}{\nu}\right\}Q^{t-1} \leqslant \left(1+\frac{1}{\nu!}\right)(2k)^{\nu-1}kQ^{t-1}. \end{aligned} \tag{19}$$

By a well-known theorem on symmetric functions, however, we have

$$s_t-\sigma_1 s_{t-1}+\sigma_2 s_{t-2}-\ldots+(-1)^t t\sigma_t = 0 \tag{20}$$

and

$$s_t'-\sigma_1' s_{t-1}'+\sigma_2' s_{t-2}'-\ldots+(-1)^t t\sigma_t' = 0. \tag{21}$$

Combining (19), (20), and (21) we obtain

$$\begin{aligned}|\sigma_t-\sigma_t'| &\leqslant \frac{1}{t}\left(1+2k+\tfrac{3}{2}k\sum_{\nu=2}^{t-1}(2k)^{\nu-1}\right)Q^{t-1} \\ &\leqslant \tfrac{1}{2}[1+2k+\tfrac{3}{2}k\{(2k)^{t-1}-2k\}/(2k-1)]Q^{t-1} \\ &\leqslant \tfrac{1}{2}\{1+\tfrac{1}{2}k+\tfrac{3}{2}k(2k)^{t-1}/(2k-1)\}Q^{t-1} \\ &\leqslant \tfrac{3}{4}(2k)^{t-1}Q^{t-1}. \end{aligned} \tag{22}$$

Consequently, we have, for $|X| \leqslant Q$, that

$$\begin{aligned}|(X-x_1)\ldots(X-x_k)-(X-y_1)\ldots(X-y_k)| &\leqslant \sum_{h=1}^{k}|\sigma_h-\sigma_h'||X|^{k-h} \\ &\leqslant \left\{1+\tfrac{3}{4}\sum_{h=2}^{k}(2k)^{h-1}\right\}Q^{k-1} \\ &\leqslant \left\{1+\frac{6k}{4(2k-1)}\{(2k)^{k-1}-1\}\right\}Q^{k-1} \\ &\leqslant (2kQ)^{k-1}, \end{aligned} \tag{23}$$

since $2k/(2k-1) \leqslant \frac{4}{3}$.

But $|y_k-x_\nu| \geqslant R$ for $\nu = 1, 2,\ldots, k-1$, so, if we set $X = y_k$ in (23), we obtain

$$R^{k-1}|y_k-x_k| \leqslant (2kQ)^{k-1}.$$

Therefore the number of x_k satisfying the requirements of our theorem does not exceed $(2kQ)^{k-1}$. Next, for fixed x_k, the numbers

$$x_1^h+\ldots+x_{k-1}^h \quad (1 \leqslant h \leqslant k-1) \tag{24}$$

lie in intervals of lengths at most Q^{h-1} $(1 \leqslant h \leqslant k-1)$ respectively. This reduces to the exact formulation of our lemma with $k-1$ instead of k. The lemma is evident for $k = 1$. We suppose that it holds for smaller k; then the number of sets of integers $x_1,..., x_{k-1}$ satisfying the requirements imposed on (24) does not exceed

$$\{2(k-1)H\}^{\frac{1}{2}(k-1)(k-2)}.$$

Therefore the number of sets of integers satisfying the requirements imposed on (11) is less than or equal to

$$\{2(k-1)H\}^{\frac{1}{2}(k-1)(k-2)}(2kH)^{k-1} \leqslant (2kH)^{\frac{1}{2}k(k-1)}.$$

LEMMA 2. *Let $c \geqslant 1$. Under the same hypothesis as in Lemma 1, the number of sets of integers $x_1,..., x_k$ for which*

$$x_1^h+...+x_k^h \quad (1 \leqslant h \leqslant k)$$

lies in intervals of lengths not exceeding $cQ^{(1-1/k)h}$ respectively $(1 \leqslant h \leqslant k)$ does not exceed

$$(2c)^k(2kH)^{\frac{1}{2}k(k-1)}Q^{\frac{1}{2}(k-1)}. \tag{25}$$

Proof. We divide the hth interval into

$$\{cQ^{h(1-1/k)}/Q^{h-1}\}+1$$

parts and apply Lemma 1. Since

$$\prod_{h=1}^{k}\{(cQ^{h(1-1/k)}/Q^{h-1})+1\} \leqslant \prod_{h=1}^{k}(2cQ^{h(1-1/k)-(h-1)}) = (2c)^kQ^{\frac{1}{2}(k-1)},$$

we have at most $(2c)^kQ^{\frac{1}{2}(k-1)}$ sets of sub-intervals, each of them satisfying the hypothesis of Lemma 1. Therefore we have at most $(2kH)^{\frac{1}{2}k(k-1)}$ solutions for each set, and the theorem follows.

LEMMA 3. *The set of integers $(g_1,..., g_b)$ with $1 \leqslant g_\nu \leqslant H$ is said to be 'well-spaced' if there are at least k of them, say $g_{j_1},..., g_{j_k}$, satisfying*

$$g_{j_{\nu+1}}-g_{j_\nu} > 1 \quad (1 \leqslant \nu \leqslant k-1). \tag{26}$$

The number of not well-spaced sets is at most

$$b!\,3^bH^{k-1}. \tag{27}$$

Proof. We arrange $g_1,..., g_b$ in order of increasing magnitude

$$1 \leqslant g_1' \leqslant g_2' \leqslant ... \leqslant g_b', \tag{28}$$

and set $f_\nu = g'_{\nu+1}-g'_\nu$. If the set is not well-spaced, there are at most $k-2$ of the f's for which $f_\nu > 1$.

Consider now these sets with exactly σ $(0 \leqslant \sigma \leqslant k-2)$ f's with $f_\nu > 1$. The number of different positions of these σ f's is $\binom{b-1}{\sigma}$. Thus the number of different sets is at most

$$\binom{b-1}{\sigma} H^{\sigma+1} 2^{b-1-\sigma}$$

since $0 \leqslant f_\nu \leqslant H-1$ and $1 \leqslant g_1' \leqslant H$. The total number of not well-spaced sets is therefore

$$\leqslant \sum_{\sigma=0}^{k-2} \binom{b-1}{\sigma} H^{\sigma+1} 2^{b-1-\sigma} \leqslant (1+2)^{b-1} H^{k-1} \leqslant 3^b H^{k-1}.$$

The theorem now follows since the number of sets $(g_1,\ldots,g_b)$ corresponding to $(g_1',\ldots,g_b')$ is $b!$.

3. Recurrence formula

THEOREM 2. *Let b be an integer $\geqslant \frac{1}{2}k(k+1)+k$ and let η be the greatest integer not exceeding*

$$\frac{1}{k}\log Q/\log 2. \tag{29}$$

Then

$$\int_0^1 \cdots \int_0^1 |C_k(Q)|^{2b}\, d\alpha_1 \ldots d\alpha_k \leqslant (7b)^{4b} \max(1,\eta^2) Q^{2k-\frac{1}{2}(k+1)+2(b-k)/k} \times$$
$$\times \int_0^1 \cdots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k. \tag{30}$$

Proof. (i) We defined

$$C_k(Q) = \sum_{T<x\leqslant T+Q} e^{2\pi i f(x)}$$

in Theorem 1. From the remarks there, we see that without loss of generality we may assume that $T = 0$ hereafter. Suppose that $\eta \geqslant 2$, and let s be an integer satisfying $1 \leqslant s \leqslant \eta-1$. We divide $C_k(Q)$ into 2^s parts, each of length $R_s = Q\,2^{-s}$:

$$C_k(Q) = \sum_{g=1}^{2^s} \sum_{(g-1)R_s<x\leqslant gR_s} e^{2\pi i f(x)}$$
$$= \sum_{g=1}^{2^s} Z_{sg}, \quad \text{say.}$$

Let $Z = \{C_k(Q)\}^b$. Then

$$Z = \sum^{2^{sb}} Z_{sg_1} \ldots Z_{sg_b}, \tag{31}$$

where $\sum\limits^{M}$ denotes a sum of at most M terms (I shall use this convention throughout the paper). I use the further abbreviation

$$Z_s = Z_{s;g_1,\dots,g_b} = Z_{sg_1}\dots Z_{sg_b}.$$

Those $Z_{s;g_1,\dots,g_b}$ with well-spaced $g_1,\dots, g_b$ are called *well-spaced sums* and are denoted by Z'_s. By Lemma 3, the number of not well-spaced sums does not exceed $b!\,3^b 2^{s(k-1)}$. Those Z_s which are not well-spaced sums are now decomposed further by dividing each factor into two parts, so that from each not well-spaced sum Z_s we obtain 2^b sums of the type Z_{s+1}. The number of well-spaced Z_{s+1} obtained from all of the not well-spaced Z_s clearly does not exceed

$$b!\,3^b 2^{s(k-1)}.2^b = b!\,6^b 2^{s(k-1)}.$$

Those Z_{s+1} thus obtained which are well-spaced are denoted by Z'_{s+1}, and we decompose the others as above. Since these Z_1 are always not well-spaced, we have no difficulty to begin with. We repeat this process for $s = 1, 2,\dots, \eta-1$, and use Z'_η to denote *all* those Z_η obtained from those not well-spaced $Z_{\eta-1}$. We have therefore

$$Z = \sum_{s=1}^{\eta} \sum^{M_s} Z'_s, \tag{32}$$

where $M_s = b!\,6^b 2^{s(k-1)}$.

(ii) By Schwarz's inequality, we obtain

$$|C(Q)|^{2b} = |Z|^2 \leqslant \eta \sum_{s=1}^{\eta} \Big|\sum^{M_s} Z'_s\Big|^2 \leqslant \eta \sum_{s=1}^{\eta} M_s \sum^{M_s} |Z'_s|^2. \tag{33}$$

Suppose $g_1,\dots, g_k$ of $Z'_{s;g_1\dots g_b}$ $(1 \leqslant s \leqslant \eta-1)$ satisfy (9); otherwise, we can rearrange the subscripts. Since the geometrical mean does not exceed the arithmetical mean, we have

$$|Z_{sg_{k+1}}\dots Z_{sg_b}|^2 \leqslant \frac{1}{b-k} \sum_{i=k+1}^{b} |Z_{sg_i}|^{2(b-k)}. \tag{34}$$

We divide Z_{sg_i} $(k+1 \leqslant i \leqslant b)$ into

$$[Q\,2^{-s}/(Q^{1-1/k}-1)]+1 \leqslant Q\,2^{-s}(Q^{1-1/k}-1)^{-1}+Q^{1/k}2^{-\eta}$$
$$\leqslant Q\,2^{-s}(\tfrac{3}{4}Q^{1-1/k})^{-1}+Q^{1/k}2^{-s-1} \leqslant Q^{1/k}2^{1-s}$$

(since $4 \leqslant 2^\eta \leqslant Q^{1/k} \leqslant Q^{1-1/k}$) parts, each of the form

$$C^* = \sum_x e^{2\pi i f(x)},$$

where x runs over an interval of length $\leqslant Q^{1-1/k}-1$; namely, we have an integer ω such that

$$\omega < x \leqslant \omega+Q', \qquad 0 < Q' \leqslant Q^{1-1/k}, \qquad 0 \leqslant \omega \leqslant g_i R_s \leqslant Q.$$

Then, by Hölder's inequality,

$$|Z_{sg_i}|^{2(b-k)} \leqslant \Big(\sum^{Q^{1/k}2^{1-s}} |C^*|\Big)^{2(b-k)} \leqslant (Q^{1/k}2^{1-s})^{2(b-k)-1} \sum^{Q^{1/k}2^{1-s}} |C^*|^{2(b-k)}. \tag{35}$$

From (33), (34), and (35) we then obtain

$$|Z|^2 \leqslant \frac{\eta}{b-k} \sum_{s=1}^{\eta} M_s (Q^{1/k}2^{1-s})^{2(b-k)-1} \sum^{N_s} |Z_{sg_1}|^2 \ldots |Z_{sg_k}|^2 |C^*|^{2(b-k)}, \tag{36}$$

where $N_s = M_s(b-k)Q^{1/k}2^{1-s} = b!\,6^b.2^{s(k-1)}(b-k)Q^{1/k}2^{1-s}$. Integrating over the unit hypercube $(0 \leqslant \alpha_1 \leqslant 1,\ldots, 0 \leqslant \alpha_k \leqslant 1)$, we have

$$\int_0^1 \ldots \int_0^1 |Z|^2\, d\alpha_1 \ldots d\alpha_k \leqslant \frac{\eta}{b-k} \sum_{s=1}^{\eta} M_s (Q^{1/k}2^{1-s})^{2(b-k)-1} \times$$

$$\times \sum^{N_s} \int_0^1 \ldots \int_0^1 |Z_{sg_1}|^2 \ldots |Z_{sg_k}|^2 |C^*|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k. \tag{37}$$

(iii) The expression

$$\int_0^1 \ldots \int_0^1 |Z_{sg_1}|^2 \ldots |Z_{sg_k}|^2 |C^*|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k \tag{38}$$

is equal to the number of solutions of the system of Diophantine equations

$$x_1^h + \ldots + x_k^h + y_1^h + \ldots + y_{b-k}^h = x_1'^h + \ldots + x_k'^h + y_1'^h + \ldots + y_{b-k}'^h \quad (1 \leqslant h \leqslant k),$$

where the y's lie in an interval of the form

$$\omega < y,\, y' \leqslant \omega + Q' \quad (0 < Q' \leqslant Q^{1-1/k};\ 0 \leqslant \omega \leqslant Q),$$

and the x_i and x_i' lie in intervals

$$(g_i - 1)R_s < x_i,\, x_i' \leqslant g_i R_s,$$

where, for $s \leqslant \eta - 1$, the integers $g_1,\ldots, g_k$ satisfy the condition (9).

We replace x by $X + \omega$ and y by $Y + \omega$. Then (38) is also the number of solutions of the system of equations

$$X_1^h + \ldots + X_k^h + Y_1^h + \ldots + Y_{b-k}^h = X_1'^h + \ldots + X_k'^h + Y_1'^h + \ldots + Y_{b-k}'^h \quad (1 \leqslant h \leqslant k), \tag{39}$$

where the Y's lie in the interval $(0, Q')$ and X_i and X_i' lie in

$$-\omega + (g_i - 1)R_s < X_i,\, X_i' \leqslant -\omega + g_i R_s \quad (0 \leqslant \omega \leqslant Q). \tag{40}$$

If now the X' are fixed arbitrarily, the conditions on the X satisfy the requirements of Lemma 1 with $R = R_s$ and Lemma 2 with $c = 2(b-k)$ and $H = 2^s$. Thus the number of sets of X and X' does not exceed

$$R_s^k\{4(b-k)\}^k(2k\,2^s)^{\frac{1}{2}k(k-1)}Q^{\frac{1}{2}(k-1)}$$
$$= \{4(b-k)\}^k(2k)^{\frac{1}{2}k(k-1)}2^{\frac{1}{2}sk(k-1)-sk}Q^{2k-\frac{1}{2}(k+1)}. \quad (41)$$

Further, for a fixed set of X and X', the number of sets of Y and Y' does not exceed

$$\int_0^1 \cdots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k,$$

since
$$\left|\int_0^1 f(x)e^{ixy}\,dx\right| \leqslant \int_0^1 |f(x)|\,dx.$$

Therefore, we have

$$\int_0^1 \cdots \int_0^1 |Z_{sg_1} \ldots Z_{sg_k}|^2 |C^*|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$
$$\leqslant \{4(b-k)\}^k(2k)^{\frac{1}{2}k(k-1)}2^{\frac{1}{2}sk(k+1)-2sk}Q^{2k-\frac{1}{2}(k+1)} \times$$
$$\times \int_0^1 \cdots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k, \quad (42)$$

for $1 \leqslant s \leqslant \eta-1$.

For $s = \eta$, we use the trivial inequality

$$\int_0^1 \cdots \int_0^1 |Z_{sg_1} \ldots Z_{sg_k}|^2 |C^*|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$
$$\leqslant R_\eta^{2k} \int_0^1 \cdots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k. \quad (43)$$

Since
$$R_\eta^{2k} = Q^{2k}2^{-2k\eta}$$
$$\leqslant 2^{-\eta[2k-\frac{1}{2}k(k+1)]}Q^{2k-\frac{1}{2}(k+1)}(Q\,2^{-\eta k})^{\frac{1}{2}(k+1)}$$
$$\leqslant 2^{-\eta[2k-\frac{1}{2}k(k+1)]}Q^{2k-\frac{1}{2}(k+1)}2^{\frac{1}{2}k(k+1)},\dagger$$

then (42) holds also for $s = \eta$.

† Since $\eta \geqslant \log Q/k \log 2 - 1$, $\log 2^\eta \geqslant \log Q^{1/k} - \log 2 = \log \frac{1}{2}Q^{1/k}$, then $Q\,2^{-\eta k} \leqslant 2^k$.

(iv) Combining (37) and (42) with $s = 1,\ldots, \eta$, we have

$$\int_0^1 \ldots \int_0^1 |C(Q)|^{2b}\, d\alpha_1 \ldots d\alpha_k \leqslant \eta \sum_{s=1}^{\eta} M_s(Q^{1/k}2^{1-s})^{2(b-k)-1} N_s\{4(b-k)\}^k \times$$

$$\times (2k)^{\frac{1}{2}k(k-1)} 2^{\frac{1}{2}sk(k+1)-2sk} Q^{2k-\frac{1}{2}(k+1)} \int_0^1 \ldots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$

$$\leqslant \eta c \sum_{s=1}^{\eta} 2^{-s\{2b-\frac{1}{2}k(k+1)-2k\}} Q^{2k-\frac{1}{2}(k+1)+2(b-k)/k} \times$$

$$\times \int_0^1 \ldots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$

$$\leqslant \eta^2 c Q^{2k-\frac{1}{2}(k+1)+2(b-k)/k} \int_0^1 \ldots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k, \tag{44}$$

since $2b \geqslant \frac{1}{2}k(k+1)+2k$, where

$$c = (b!\, 6^b)^2 2^{2(b-k)} (4b)^k (2k)^{\frac{1}{2}k(k-1)}.$$

Since $\quad c < (12b)^{2b}(4b)^b(2k)^b \leqslant \{(12b)^2 . 4b . 2b\}^b \leqslant (7b)^{4b},$

we have the theorem for $\eta \geqslant 2$.

(v) *The case* $\eta < 2$. Then

$$\frac{1}{k}\log Q/\log 2 < 2, \quad \text{i.e.} \quad Q < 4^k.$$

We divide $C_k(Q)$ into four parts, each of them of the form

$$C^* = \sum_{\omega < x < \omega+Q'} e^{2\pi i f(x)} \quad (0 < Q' \leqslant \tfrac{1}{4}Q \leqslant Q^{1-1/k}).$$

By Hölder's inequality, we have

$$|C_k(Q)|^{2b} \leqslant 4^{2b-1} \sum^4 |C^*|^{2b} \leqslant 4^{2b-1} Q^{2k(1-1/k)} \sum^4 |C^*|^{2(b-k)}.$$

Integrating over the unit hypercube, we have

$$\int_0^1 \ldots \int_0^1 |C_k(Q)|^{2b}\, d\alpha_1 \ldots d\alpha_k$$

$$\leqslant 4^{2b-1} Q^{2k(1-1/k)} \sum^4 \int_0^1 \ldots \int_0^1 |C^*|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$

$$\leqslant 4^{2b} Q^{2k(1-1/k)} \int_0^1 \ldots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k$$

$$\leqslant 4^{2b} Q^{2b-\frac{1}{2}(k+1)+2(b-k)/k} \int_0^1 \ldots \int_0^1 |C_k(Q^{1-1/k})|^{2(b-k)}\, d\alpha_1 \ldots d\alpha_k,$$

since $2b > \frac{1}{2}k(k+1)$, and we have the theorem.

4. Proof of Theorem 1. If $P^{1-1/k} \leqslant 3$, then $P \leqslant 9$, and the theorem is trivial. Accordingly we assume that $P^{1-1/k} > 3$, and consequently $P > e$.

The theorem is trivial for $l = 0$. We use induction on l, and we assume that it is true for $l-1$. By Theorem 2, we have

$$\int_0^1 \cdots \int_0^1 |C_k(P)|^{2s}\, d\alpha_1 \ldots d\alpha_k \leqslant (7s)^{4s} P^{2k-\frac{1}{2}(k+1)+2(s-k)/k} \times$$
$$\times (\log P)^2 \int_0^1 \cdots \int_0^1 |C_k(P^{1-1/k})|^{2(s-k)}\, d\alpha_1 \ldots d\alpha_k. \qquad (45)$$

By the inductive hypothesis, with $l-1$, $s-k$, and $P^{1-1/k}$ instead of l, s, and P in the statement of Theorem 1, we have, for $P^{1-1/k} > 3 > 2$,

$$\int_0^1 \cdots \int_0^1 |C_k(P^{1-1/k})|^{2(s-k)}\, d\alpha_1 \ldots d\alpha_k \leqslant (7s)^{4s(l-1)} (\log P)^{2(l-1)} \times$$
$$\times P^{(1-1/k)\{2s-2k-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)(1-1/k)^{l-1}\}}. \qquad (46)$$

Combining (45) and (46), we have the theorem.

As a consequence of Theorem 1, we have

THEOREM 3. *Let $P \geqslant 2$ and $s \geqslant \frac{1}{4}k(k+1)+lk$, then we have*

$$\int_0^1 \Big| \sum_{x=1}^{P} e^{2\pi i \alpha x^k} \Big|^{2s} d\alpha_1 \leqslant s^k (7s)^{4sl} (\log P)^{2l} P^{2s-k+\delta},$$

where $\delta = \frac{1}{2}k(k+1)(1-1/k)^l$.

Proof. Let $r(N_1,\ldots,N_k)$ be the number of solutions of

$$x_1^h + \ldots + x_s^h - y_1^h - \ldots - y_s^h = N_h \quad (1 \leqslant h \leqslant k;\ 1 \leqslant x, y \leqslant P).$$

Evidently, we have

$$\int_0^1 \Big| \sum_{x=1}^{P} e^{2\pi i \alpha x^k} \Big|^{2s} d\alpha \leqslant \sum_{|N_1| < sP} \cdots \sum_{|N_{k-1}| < sP^{k-1}} r(N_1,\ldots,N_{k-1}, 0),$$

since

$$r(N_1,\ldots,N_{k-1},N_k) = \int_0^1 \cdots \int_0^1 |C_k(P)|^{2s} e^{2\pi i (N_1\alpha_1+\ldots+N_k\alpha_k)}\, d\alpha_1 \ldots d\alpha_k$$
$$\leqslant \int_0^1 \cdots \int_0^1 |C_k(P)|^{2s}\, d\alpha_1 \ldots d\alpha_k.$$

The theorem is therefore an immediate consequence of Theorem 1.

5. Estimation of exponential sums

LEMMA 4. *Let* $$\left|\alpha-\frac{h}{q}\right| \leqslant \frac{1}{q^2}, \quad (h,q)=1.$$

Then $$\sum_{y=1}^{Y}\left|\sum_{n=f+1}^{f+N} e^{2\pi i\alpha yn}\right| \leqslant \left(\frac{Y}{q}+1\right)(N+q\log q).$$

The lemma is well known; for a proof see, for instance, Landau (**2**).

THEOREM 4. *Suppose that the number of solutions of the system of Diophantine equations*

$$x_1^h+\ldots+x_{2t}^h = y_1^h+\ldots+y_{2t}^h \quad (1 \leqslant h \leqslant k;\ 1 \leqslant x,y \leqslant Q) \tag{47}$$

does not exceed $$c_1(k,t)P^{4t-\frac{1}{2}k(k+1)+\delta'}. \tag{48}$$

Let $F(x) = \alpha_{k+1}x^{k+1}+\ldots+\alpha_1 x+\alpha_0$, *and*

$$S = \sum_{x=1}^{P} e^{2\pi i F(x)}, \tag{49}$$

and $$|\alpha_{k+1}-h/q| \leqslant q^{-2}, \quad (h,q)=1, \quad P \leqslant q \leqslant P^k.$$

Then we have

$$|S| \leqslant c_2(k,t)P^{1-\rho}, \qquad \rho = (1-\delta)/(4t+k-\delta). \tag{50}$$

Proof. Let p_1 be an integer, $1 \leqslant p_1 \leqslant P$, and let

$$S(y) = \sum_{x=1}^{p_1} e^{2\pi i F(x+y)}.$$

Then
$$S = \frac{1}{p_1}\sum_{x=1}^{p_1}\sum_{z=1}^{P} e^{2\pi i F(z)} = \frac{1}{p_1}\sum_{x=1}^{p_1}\sum_{y=1-x}^{P-x} e^{2\pi i F(x+y)}$$

$$= \frac{1}{p_1}\sum_{y=1}^{P} S(y)+Q_1 p_1, \quad \text{where} \quad |Q_1| \leqslant 1. \tag{51}$$

Write $$F(x+y) = A_{k+1}x^{k+1}+A_k x^k+\ldots+A_0.$$

Then $$A_{k+1} = \alpha_{k+1}, \qquad A_k = \alpha_k+(k+1)\alpha_{k+1}y, \qquad \ldots. \tag{52}$$

By Hölder's inequality, we have

$$\left|\sum_{y=1}^{P} S(y)\right|^{2t} \leqslant P^{2t-1}\sum_{y=1}^{P}|S(y)|^{2t} = P^{2t-1}\sum_{y=1}^{P}\{S(y)\}^t\{\overline{S(y)}\}^t$$
$$= P^{2t-1}\sum_{y=1}^{P}\left(\sum_{x_1=1}^{p_1}\ldots\sum_{x_t=1}^{p_1}\sum_{x_1'=1}^{p_1}\ldots\sum_{x_t'=1}^{p_1} e^{2\pi i\phi}\right), \tag{53}$$

where

$$\phi = f(x_1+y)+\ldots+f(x_t+y)-f(x_1'+y)-\ldots-f(x_t'+y)$$
$$= A_{k+1}\left(\sum_{i=1}^{t} x_i^{k+1}-\sum_{i=1}^{t} x_i'^{k+1}\right)+A_k\left(\sum_{i=1}^{t} x_i^k-\sum_{i=1}^{t} x_i'^k\right)+\ldots.$$

Let $\psi(N_k,\ldots,N_1)$ be the number of solutions of

$$x_1^h+\ldots+x_t^h-x_1'^h-\ldots-x_t'^h = N_h \quad (1 \leqslant h \leqslant k;\ 1 \leqslant x,x' \leqslant p_1).$$

Then we have

$$\sum_{y=1}^{P} |S(y)|^{2t} \leqslant \sum_{x_1=1}^{p_1} \cdots \sum_{x_t=1}^{p_1} \sum_{x_1'=1}^{p_1} \cdots \sum_{x_t'=1}^{p_1} \Big| \sum_{y=1}^{P} e^{2\pi i \phi} \Big|$$

$$\leqslant \sum_{|N_1|<tp_1} \cdots \sum_{|N_k|<tp_1^k} \psi(N_k,\ldots,N_1) \Big| \sum_y \exp(A_k N_k+\ldots+A_1 N_1) \Big|$$

$$\leqslant \sqrt{\Big\{ \sum_{N_1} \cdots \sum_{N_k} \psi^2(N_k,\ldots,N_1) \sum_{N_1} \cdots \sum_{N_k} \Big| \sum_y \exp(A_k N_k+\ldots+A_1 N_1) \Big|^2 \Big\}} \tag{54}$$

by Schwarz's inequality.

First the expression

$$\sum_{N_1} \cdots \sum_{N_k} \psi^2(N_k,\ldots,N_1) = \sum_{N_1} \cdots \sum_{N_k} \Big| \int_0^1 \cdots \int_0^1 \Big| \sum_{x=1}^{p_1} \exp(\alpha_k x^k+\ldots+\alpha_1 x) \Big|^{2t} \times$$

$$\times \exp(-N_k \alpha_k-\ldots-N_1 \alpha_1)\, d\alpha_1\ldots d\alpha_k \Big|$$

$$= \int_0^1 \cdots \int_0^1 \Big| \sum_{x=1}^{p_1} \exp(\alpha_k x^k+\ldots+\alpha_1 x) \Big|^{4t} d\alpha_1\ldots d\alpha_k,$$

by the Parseval relation. By (48), we have

$$\sum_{N_1} \cdots \sum_{N_k} r^2(N_1,\ldots,N_k) \leqslant c_1(k,t) p_1^{4t-\frac{1}{2}k(k+1)+\delta'}. \tag{55}$$

Next, we have, by (52),

$$\sum_{|N_1|<tp_1} \cdots \sum_{|N_k|<tp_1^k} \Big| \sum_y e^{2\pi i(A_k N_k+\ldots+A_1 N_1)} \Big|^2$$

$$\leqslant t^k p_1^{1+\ldots+k-1} \sum_{y_1=1}^{P} \sum_{y_2=1}^{P} \Big| \sum_{N_k} e^{2\pi i(k+1)\alpha_{k+1}(y_1-y_2)N_k} \Big|$$

$$\leqslant t^k p_1^{\frac{1}{2}k(k-1)} P \sum_{Y=1}^{P} \Big| \sum_{N_k} e^{2\pi i \alpha_{k+1} Y N_k} \Big|,$$

(since the number of solutions of $(k+1)(y_1-y_2) = Y$ does not exceed P)

$$\leqslant c_3(k,t) p_1^{\frac{1}{2}k(k-1)} P \Big(\frac{P}{q}+1\Big)(p_1^k+q\log q) \tag{56}$$

by Lemma 4.

Combining (54), (55), (56) we have

$$\sum_y |S(y)|^{2t} \leqslant c_4(k,t) p_1^{2t+\frac{1}{2}\delta} P^{\frac{1}{2}} (1+p_1^{-k} q \log q)^{\frac{1}{2}}.$$

Consequently, from (53), we deduce

$$|S| \leqslant c_5(k,t) p_1^{\delta/4t} P^{1-1/4t} (1+p_1^{-k} q \log q)^{1/4t} + p_1.$$

For $P \leqslant q \leqslant P^k$, we have

$$|S| \leqslant c_5(k,t)(p_1^{(\delta-k)/4t} P^{1+(k-1)/4t} + p_1)\log P.$$

Taking
$$p_1 = P^{1-\rho}, \qquad \rho = \frac{1-\delta}{4t+k-\delta},$$
we have the theorem.

THEOREM 5. *Let $k > 10$, and*

$$\left|\alpha - \frac{h}{q}\right| \leqslant \frac{1}{q^2}, \quad (h,q) = 1, \quad P \leqslant q \leqslant P^{k-1},$$

then

$$\left|\sum_{x=1}^{P} e^{2\pi i \alpha x^k}\right| \leqslant c_6(k) P^{1-1/\sigma}, \quad \sigma = 4k^2(\log k + \tfrac{1}{2}\log\log k + 3).$$

Proof. Taking

$$t = \left[\tfrac{1}{8}k(k-1) + \frac{l(k-1)}{2}\right] + 1, \qquad l = \left[\frac{\log\{\frac{1}{2}k(k-1)\} + \log\log k}{-\log\{1-1/(k-1)\}}\right] + 1;$$

we have
$$\delta = \tfrac{1}{2}k(k-1)\left(1 - \frac{1}{k-1}\right)^l < 1/\log k \quad (< \tfrac{1}{2}).$$

By Theorem 1, the hypotheses of Theorem 4 are true, with $\delta' = 1/\log k$.

Since
$$-1\Big/\log\left(1 - \frac{1}{k-1}\right) \leqslant k,$$

$$2l + \tfrac{1}{2}k \leqslant 2k(\log \tfrac{1}{2}h^2 + \log\log k) + \tfrac{1}{2}k + 1 < 2k(\log k^2 + \log\log k),$$

then we have

$$\begin{aligned} \frac{4t-k-1-\delta}{1-\delta} &\leqslant \frac{(2l+\frac{1}{2}k)(k-1)-k+3-\delta}{1-\delta} \\ &< (2l+\tfrac{1}{2}k)k(1+2\delta) \\ &\leqslant 4k^2(\log k + \tfrac{1}{2}\log\log k + 2 + \log\log k/\log k) \\ &\leqslant 4k^2(\log k + \tfrac{1}{2}\log\log k + 3). \end{aligned} \tag{57}$$

The theorem follows from Theorem 4, since (57) is an open sign.

6. Applications

In this section I shall only indicate several applications which require merely straightforward alternation of the known methods.

(i) *The Waring–Goldbach problem.* Let $H(n)$ be the least integer s such that

$$p_1^k + \dots + p_s^k = N \tag{58}$$

is soluble for larger N, provided that N satisfies certain congruence

conditions. By the method used in one of the author's papers (**3**), with the new exponent of Theorem 5 instead of the old one, we have

$$H(k) \leqslant s_0 \quad (\sim 4k\log k,\ k \text{ large}).$$

(ii) The asymptotic formula for the number of solutions of (58) is true when $s \geqslant s_0 = 4k^2(\log k + \frac{1}{2}\log\log k + 8)$. Certainly, we can also prove that the Hardy–Littlewood asymptotic formula for the number of decompositions of an integer into s positive kth powers holds also for this bound, which is sharper than Vinogradov's (**4**) bound $s \geqslant s_0 = 10k^2 \log k$.

REFERENCES

1. The latest version of Vinogradov's method, see *Bull. de l'Acad. des Sci. de l'URSS*, 6 (1942), 33–40.
2. Landau, *Vorlesungen über Zahlentheorie*, Bd. 1, p. 256.
3. Hua, *Math. Zeits.* 44 (1939), 335–46.
4. Vinogradov, *Comptes Rendus* (*Doklady*).

[*Added* 20 *July* 1949.] I am indebted to Dr. J. L. B. Cooper for the information that my booklet was published in 1947 as No. 22 of the *Travaux de l'Institut math. Stekloff*. Also he showed me a copy of Vinogradov's booklet which contains the proof of the result of (ii) § 6 with $s \geqslant 10k^2 \log k$. It seems also worthy of mention that the exponent of Theorem 5 is slightly better than his result. He has

$$\sigma = 3k^2 \log\{12k(k+1)\} \quad (\sim 6k^2 \log k)$$

instead of

$$\sigma = 4k^2(\log k + \tfrac{1}{2}\log\log k + 3)$$

in this paper.

NOTE ON THE HOMOTOPY GROUPS OF SPHERES

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J. H. C. WHITEHEAD has pointed out that the formula at the top of p. 308 of my paper 'Über die Klassen der Sphärenabbildungen, I' (*Compositio Math.* 5 (1937), 299–314) is incorrect. On his suggestion I not only correct this formula but also expand the argument in which it occurs. The notations will be similar to those of my paper.

1. S denotes a sphere with a simplicial subdivision and a given orientation; the dimension is given by a superior index. The north pole $\mathfrak{n}$ of S^d is an interior point of a d-dimensional simplex; $\mathfrak{s}$ is the south pole of S^d; $u_1,\dots, u_\alpha$ are local Cartesian coordinates in S^d with origin $\mathfrak{n}$; f is a simplicial transformation, $f(S^e) \subset S^d$ $(e > d)$; $\mathfrak{m}Z$ is the original set of $\mathfrak{n}$: $\mathfrak{m}Z$ consists of a finite number of pseudo-varieties. The main point of the proof of the Hilfssatz is a homotopic change of f, so that $\mathfrak{m}Z$ is reduced to a unique pseudo-variety. I shall repeat it.

2. Given a general point p_1 of $\mathfrak{m}Z$ we introduce local Cartesian coordinates $x_1,\dots, x_e$ in S^e with the same orientation as that of S^e and with p_1 as origin, so that in these coordinates f becomes the projection

$$u_\nu = x_\nu \quad (\nu = 1,\dots, d).$$

(The condition of orientation can be fulfilled because $e > d$.)

3. Given a general point p_2 of $\mathfrak{m}Z$ we introduce local Cartesian coordinates $y_1,\dots, y_e$ in S^e with the same orientation as that of S^e and with p_2 as origin, so that in these coordinates f becomes the projection

$$u_1 = -y_1, \qquad u_\nu = y_\nu \quad (\nu = 2,\dots, d).$$

4. Given two disjoint sets E_1 and E_2 of S^e with Cartesian coordinate-systems x, y respectively of the same orientation, so that

$$E_1 = (|x_1| \leqslant \tfrac{1}{4}, |x_2| \leqslant 1,\dots, |x_e| \leqslant 1),$$

$$E_2 = (|y_1| \leqslant \tfrac{1}{4}, |y_2| \leqslant 1,\dots, |y_e| \leqslant 1),$$

and given a simple arc C joining the points $(\tfrac{1}{4}, 0,\dots, 0)$ of E_1 and $(-\tfrac{1}{4}, 0,\dots, 0)$ of E_2 and containing no other points of E_1 and E_2, we

construct a set D of S^e, containing E_1, E_2, and C, and provided with Cartesian coordinates $z_1,\dots,z_e$, so that

$$D = (|z_1| \leqslant 1,\dots, |z_e| \leqslant 1),$$

and

$$z_1 = x_1 - \tfrac{3}{4}, \qquad z_\nu = x_\nu \quad (\nu = 2,\dots,e) \quad \text{in } E_1,$$

$$z_1 = y_1 + \tfrac{3}{4}, \qquad z_\nu = y_\nu \quad (\nu = 2,\dots,e) \quad \text{in } E_2.$$

5. We take E_1 and E_2 as neighbourhoods of p_1 and p_2 satisfying the conditions of § 4 with coordinates x and y according to §§ 2, 3. We take the simple arc C so that it contains no point of $\mathfrak{m}Z$, and afterwards we take D according to § 4, and so that it contains no point of $\mathfrak{m}Z$ except those in E_1, E_2. The sub-set $z_{d+1} = \dots = z_e = 0$ of D is named D'; $(\pm\frac{3}{4}, 0,\dots, 0)$ are the only originals of $\mathfrak{n}$ in D'.

6. Let ρ be a transformation of S^d in itself, so that

(i) $\rho(q)$ is on the same meridian as q,

(ii) ρ is a homeomorphism of some small neighbourhood of $\mathfrak{n}$ on $S^d \backslash \mathfrak{s}$,

(iii) ρ transforms the remaining part of S^d in $\mathfrak{s}$.

Now $g = \rho f$ is homotopically equivalent to f. We may suppose that g takes $D\backslash(E_1 \cup E_2)$ and the boundary of D' into $\mathfrak{s}$. Then g is independent of $z_{d+1},\dots, z_e$, and $g(z_1, z_2,\dots, z_e) = g(-z_1, z_2,\dots, z_e)$ in D.

7. We define, for $0 \leqslant \tau \leqslant 1$,

$$g_\tau(z_1,\dots, z_d, 0,\dots, 0) = \begin{cases} g\{-\tau+(1-\tau)z_1, z_2,\dots, z_d, 0,\dots, 0\} & \text{if } z_1 \leqslant 0, \\ g\{\tau+(1-\tau)z_1, z_2,\dots, z_d, 0,\dots, 0\} & \text{if } z_1 \geqslant 0. \end{cases}$$

8. We define

$$g_\tau(z_1,\dots, z_d, z_{d+1},\dots, z_e) = g_{\alpha\tau}(z_1,\dots, z_d, 0,\dots, 0) \quad \text{in } D,$$

where

$$\alpha = \min(1-|z_{d+1}|,\dots, 1-|z_e|),$$

and

$$g_\tau = g \quad \text{outside of } D.$$

9. Then we have in D

$$g_1(z_1,\dots, z_e) = g_\alpha(z_1,\dots, z_d, 0,\dots, 0) = \begin{cases} g\{-\alpha+(1-\alpha)z_1, z_2,\dots, z_d, 0,\dots, 0\} & \text{if } z_1 \leqslant 0, \\ g\{\alpha+(1-\alpha)z_1, z_2,\dots, z_d, 0,\dots, 0\} & \text{if } z_1 \geqslant 0. \end{cases}$$

10. The g-original set of $\mathfrak{n}$ in D' is $(\pm\frac{3}{4}, 0,\dots, 0)$. The g_τ-original set of $\mathfrak{n}$ in D' is $\{\pm(\tau-\frac{3}{4})/(1-\tau), 0,\dots, 0\}$ if $\tau \leqslant \frac{3}{4}$, and void if $\tau > \frac{3}{4}$.

The g_1-original set of $\mathfrak{n}$, in D, is $\{\pm(\alpha-\frac{3}{4})/(1-\alpha), 0,..., 0, x_{d+1},..., x_e\}$ $(\alpha \leqslant \frac{3}{4})$. It is different from $\mathfrak{m}Z$ only in D. The two $(e-d)$-dimensional cubes $z_1 = \pm\frac{3}{4}, z_2 = ... = z_d = 0$ have been bored out, and the boundaries of these holes have been joined by a tube.

11. By means of such tubes we join one component of $\mathfrak{m}Z$ with the other components, taking the points p_1 and p_2 in different components. After this $\mathfrak{m}Z$ will have the desired form.